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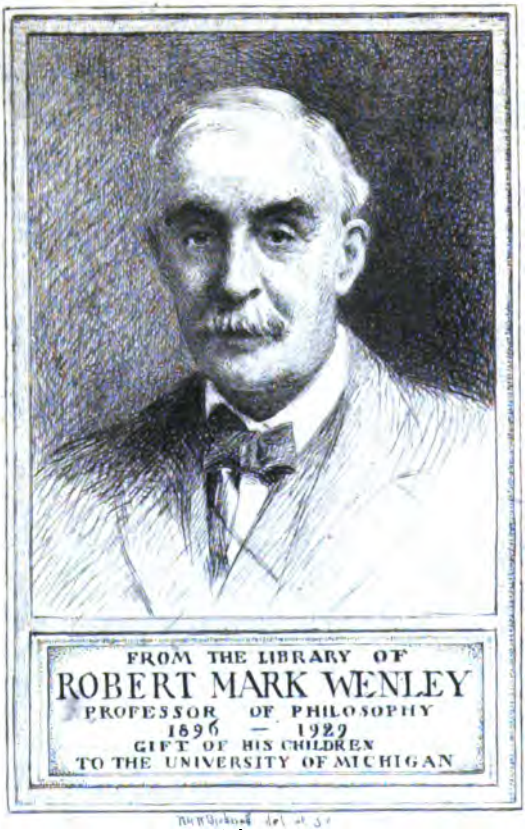

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SCHOOL ALGEBRA

BY

W. E. PATERSON, M.A., B.Sc.

MATHEMATICAL MASTER
MERCERS' SCHOOL, LONDON

PART I

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PREFACE

THE main object of this work is to provide a book on modern lines that is suitable for the beginner and can be continued in use for higher forms. In the order adopted and in general treatment the suggestions of the Mathematical Association have been largely followed.

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The early chapters are very full in explanation, and are written in simple, even colloquial, language. As far as possible, mathematical terms are not introduced until the processes to which they refer have been rendered familiar by use or are actually going to be used. The usual preliminary list of technical words and their definitions is avoided by placing a general index at the end of the book.

By means of examples, some worked in the text, others set as exercises, the student is led to discover or verify the fundamental laws of Algebra. This experimental method is continued throughout the book. Many examples are set with the intention of leading the student up to a difficulty: thus in some of the early examples such expressions as $3-5$ occur before negative quantities have been mentioned. These examples show the necessity for the next step in the subject, and by giving the student the chance of discovering that step for himself tend to develop the faculty of research.

The necessity of rigorous mathematical treatment is kept in view, and the usual formal proofs are given after the rules have been treated experimentally.

In the middle of Part I there is a Revision Chapter which goes over the early portions again, laying stress on the important points, and introducing new matter which was considered too difficult for the early chapters. There are also several cases of intentional repetition. For example, the importance of the formula $a^2-b^2=(a+b)(a-b)$ is insisted upon two or three times. The author thinks that repetition is more effective than mere reference to a preceding page.

In many places suggestions are made as to how the solutions to questions should be thought out, and how the work should be arranged. On such points of detail each teacher probably has his own views. But it is hoped that the insertion of such matter in a textbook will not only be of help to the inexperienced teacher, but will save the time of the experienced teacher in cases where the pupil has forgotten some previous rule or method of work.

Graphs are treated as illustrations of Algebraic principles or scientific observations, rather than as objects of importance in themselves. But to use them with skill and confidence considerable practice is necessary, and ample provision is made for this.

The practical use of Logarithms is so important that a short treatment of Fractional and Negative Indices is given in Part I, thus enabling this Part to conclude with a chapter on the use of Logarithms.

Horner's Synthetic Method of Division is introduced as an alternative to the usual method. It is surprising that this simple and compact process has not been brought into common use sooner.

The problem is regarded as the important type of example, and most of the various Algebraic operations and processes are introduced as being necessary for the solution of some practical problem. But the more mechanical type of example is not neglected: such examples afford practice in accuracy and neatness, and no student can do good work in Higher Mathematics who has not had experience in dealing with long mechanical Algebraic operations.

Many of the examples are original, but the greater number are taken from recent Examination Papers by permission of the following authorities:—

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The author's thanks are due to his colleagues, Mr. F. W. Crampton, M.A., Mr. L. Green, M.A., and Mr. C. B. Wheeler, M.A., for suggestions and help in reading the proofs.

W. E. P.

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104	Line	9.	$-4x$ *should be*		$-8x$
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Oxford and Cambridge Schools Examination Board.

JULY, 1906

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xxxvi	**No. 12.**	$(2x+1)$ *should be*	$(2x+5)$

CHAPTER I

THE USE OF LETTERS AND SIGNS

1. CONSIDER the following question:—

If a farmer has 13 *cows in one field,* 16 *cows in a second field, and* 5 *cows in a third, how many cows has he altogether?*

The answer is obtained by adding the three numbers 13, 16, 5, and the work might be arranged thus:—

$$\begin{array}{r} 13 \\ 16 \\ 5 \\ \hline \textit{Answer}\quad 34 \text{ cows.} \end{array}$$

If we wish to make quite clear what we have done we use the sign + to show that the number following is to be added to the number before, and the sign = instead of the word 'equals'. The work is then like this:—

$$\begin{aligned} \text{Total number of cows} &= 13+16+5 \\ &= 34 \quad \textit{Answer.} \end{aligned}$$

The sign + is called **plus**, and is read as plus. 'Six plus five' is written $6+5$; $7+2$ is read 'seven plus two'.

2. The sign − is called **minus**, and means that the number following is to be subtracted from the number before. 'Seven minus three' is written $7-3$, and means that three is to be subtracted from seven.

Examples I a.

The method of working the following examples is to be made clear by using the signs + and −.

1. A baker leaves the shop with 150 loaves. He sells 25 loaves in one street, 48 in another, and 75 in another. How many loaves did he sell and how many did he bring back?

2. There are 35 boys in a form; at the end of the term 15 boys are moved into a higher form and 5 boys leave, while 12 boys are moved up from the form below. How many boys are there in the form in the next term, 3 new boys having come at the beginning of the term?

3. A man has at the beginning of the year 115 books. During the year he buys 27 new books, lends 15, and gives away 17; he himself borrows 4 books. How many books are there in his library at the end of the year?

4. On a journey a man travels 95 miles by train, 12 miles by coach, and walks 5 miles. What was the whole length of the journey?

5. A man's age is 5 years more than his two sons' ages together. He is 55 years old, and one of the sons is 30 years old. How old is the other son?

6. On Sunday at midnight the thermometer stands at 35°; during Monday it rises 15°, it falls 12° on Tuesday, and rises 6° on Wednesday. What is the reading of the thermometer at midnight on Wednesday?

7. A man starts out with £6 in his pocket; he pays bills to the amount of £5, cashes a cheque for £10, and loses a sovereign. How much has he when he returns home?

8. There are four gallons of beer in a cask; if three gallons of water are poured in and two gallons of the mixture drawn out, how much is there then in the cask?

9. In a family of 5 brothers, the youngest is 3 years old, the next 5 years old, the third is as old as the other two together, the fourth is 10 years old, and the eldest is 15 years old. What is the sum of all the ages?

10. After 214 people had left a meeting there were 356 people left. How many were there at first?

11. At the end of a year a merchant finds that his property is worth £12055; during the year he had gained £517. What was the value of his property at the beginning of the year?

3. The question in § 1 would be done the same way whatever the actual number of cows might be in the different fields. In all cases we should find the total number by adding the several numbers. To show this we use letters instead of numbers. Instead of saying '*A farmer*

has a certain number of cows in one field, a certain number in a second field, and another number in a third field', we may say '*A farmer has* x *cows in one field,* y *cows in a second, and* z *cows in a third'*.

If we are then asked how many cows are there altogether, the answer is $x+y+z$. This means that when we know what numbers x, y, z stand for, we are to add those numbers.

In many questions some of the numbers may have to be added and some to be subtracted, and in such cases care must be taken that the proper sign + or − is placed before each quantity. Quantities connected by the signs + or − are called terms.

4. In working examples it is not only necessary to get the right answer but also to show quite clearly how the answer is obtained.

All work must be shown; no work should be done on any other piece of paper.

Make it quite clear what the answer is.

Example I. *A dealer has* 1000 *sacks of potatoes for sale; he takes orders for* 215, 332, 120, *and* 426 *sacks. How many more sacks must he buy in order to carry out his orders?*

The dealer takes orders for $215+332+120+426$ sacks.

Total number of sacks required = 1093.

He has 1000 sacks.

$$\begin{array}{r} 215 \\ 332 \\ 120 \\ 426 \\ \hline 1093 \end{array}$$

$\therefore$ Number of sacks he must buy $= 1093-1000.$

$= 93$ *Answer.*

Example II. *A dealer has a stock of* x *sacks of potatoes; he takes orders for* a, b, *and* c *sacks. How many sacks will he have left?*

Number of sacks left $= x-a-b-c.$

Examples I b.

1. A boy has 6*d.* in one pocket and 9*d.* in his money-box; he buys a book for 10*d.* How much has he left?

2. Write out the previous question, using x, y, z instead of 6, 9, 10. What is the answer now?

3. A man walks x miles due east, then y miles west; how far is he now from his starting-point? What distance has he walked altogether?

4. What are the answers to example 3 when $x = 7$, $y = 4$?

5. Make up a question to which the answer shall be $x-y+z$.

6. Find the value of $p-q+r$ when

(i) $p = 3$, $q = 2$, $r = 5$; (iii) $p = 23$, $q = 14$, $r = 0$;
(ii) $p = 13$, $q = 21$, $r = 25$; (iv) $p = 114$, $q = 57$, $r = 178$.

7. When $x = 12$, $y = 3$, $z = 5$, find the value of

(i) $x+y+z$; (iii) $x+y-z$;
(ii) $x-y+z$; (iv) $x-y-z$.

8. There are three forms in a school; in the first form there are x boys, in the second y boys, and there are p boys in the school altogether; how many boys are there in the third form? What does the answer become when $p = 57$, $x = 15$, $y = 21$?

9. Two men form a partnership; one put £x into the business, the other £y. During the first year they spend £x and receive £700. How much have they at the end of the year?

10. An article weighs x lb.; when it is being weighed it is found that a 10 lb. weight will not counterbalance it. What additional weight must be placed with the 10 lb. weight so that there may be an exact balance?

11. If in question 10 it were found that the 10 lb. weight was too much, what weight would have to be added to the other side to get the exact balance?

12. An article, weighing x lb., together with a weight of 4 lb. exactly counterbalances two weights of 10 and 2 lb. What is the value of x?

13. A fish weighed 12 lb.; the head weighed x lb., the body 6 lb.; what did the tail weigh?

14. A field is x yards long and y yards wide; what length of fence is required to surround it?

15. What length of tiling must be used to surround a bed of flowers which is x yards wide and twice as long?

5. Signs of Multiplication.

Example I. *If each man in a cricket eleven makes* 8 *runs and there are* 8 *extras, what is the total for the innings?*

Here there are twelve 8's to be added; in other words the total is twelve times 8, which equals 96.

The process of finding the sum of a number of equal quantities is called **multiplication**, and is usually denoted in Arithmetic by the symbol $\times$, and sometimes by a full stop, e.g. 7×8, 7.8. These both mean 7 multiplied by 8 or 7 times 8. The full stop must not be confused with a decimal point, which is placed higher up. The result of multiplication is called the **product**

Example II. *If each man in a cricket eleven makes* x *runs and there are* x *extras, what is the total for the innings?*

The total is twelve times x, and is written $12x$.

Thus in Algebra we need use no sign for multiplication, but may simply write the quantities one after the other. If one of the quantities is a number and the other a letter, then the number is called the **coefficient** of the letter; thus in $7x$, 7 is the coefficient of x.

If both the quantities are letters, either of them may be called the **coefficient** of the other.

Notice that the letter x, or a, or y, or any single letter, may be regarded as having coefficient 1, i. e. x is the same as $1x$. But the coefficient 1 is never actually written; it is always left out or understood.

Quantities which are multiplied together are called **factors** of the result. Thus in $9abc$ the factors are 9, a, b, c; but in $a+b+c$, a, b, and c are terms (§ 8).

Multiplication by 0.

Example III. *If no pennies are given to each of* 100 *boys, how much money is given away?*

Method I. Each boy receives 0 pence.

$\therefore$ 100 boys receive 100×0 pence.

Method II. Each boy receives no pennies.
∴ no money is given away.
i. e. 100 boys receive no pennies.

Comparing the two answers it is seen that $100 \times 0 = 0$.

Similarly it is seen in all cases that when any number is multiplied by nothing the product is nothing. If $x = 0$, then $7x = 0$, $8x = 0$, $ax = 0$, $10000x = 0$, &c.

6. Signs for Division.

Example I. *If a sum of* 16 *shillings is divided among* 4 *people so that they have equal shares, how much will each receive?*

To find the value of one share the whole amount must be divided by 4. The sign ÷ means 'divided by', so that the work of this example can be put down thus:—

Amount each person will receive = 16*s.* ÷ 4.
= 4*s.* *Answer.*

Very often we show that one quantity is to be divided by another by placing the first over the second with a line in between; thus $\frac{16}{4}$ means 16 divided by 4.

Both signs can be used with letters, so that $\frac{a}{3}$, $b \div 3$, $\frac{x}{a}$ mean respectively a divided by 3, b divided by 3, x divided by a. Sometimes instead of $x \div a$, x/a is used.

Example II. *A man left home with £*x *in his pocket; he spent £*y*, and then divided the rest equally among* 15 *boys. How much did each receive?*

After shopping he had $x - y$ pounds left.
Hence he divided $x - y$ pounds among 15 boys.

∴ each boy received $\frac{x-y}{15}$ pounds.

Example III. *A man had £*x*; he gave £*5 *to each of his servants and then had £*y *left. How many servants had he?*

He gave away $x - y$ pounds.
He gave 5 pounds to each servant.

∴ he had $\frac{x-y}{5}$ servants.

Examples Ic.

1. What is the value of x horses at £a each?

2. If y lb. of sugar cost £x, how many pence will 1 lb. cost?

3. If 3 cheeses cost 3p shillings, what will x cheeses cost?

4. Two places, A and B, are x miles apart; how long will it take to walk from A to B at the rate of 4 miles per hour?

5. In a certain library there are two rooms; one room contains six shelves with fifty books on each shelf, whilst the other contains eight shelves with forty-two books on each shelf. How many books are there in the library?

6. In a certain library there are two rooms; one room contains x shelves with 20 books on each shelf, whilst the other contains eight shelves with y books on each shelf. How many books are there in the library?

7. What does the answer to No. 6 become in each of the following cases?

(i) $x = 5, y = 18$; (ii) $x = 20, y = 0$; (iii) $x = 30, y = 25$.

8. If $a = 1, b = 2, c = 3, d = 0$, find the value of

$$8ab + 3bc + abc - 7bcd.$$

9. A man had x horses, he kept y and sold the remainder. He received z pounds; how much did he get for each horse?

10. What does the answer to No. 9 become when

$$z = 400, x = 25, y = 5?$$

11. If $x = 3, y = 4, z = 0$, find the value of

$$5xy + \frac{8x}{y} - 9xyz.$$

12. If 1 lb. of butter costs x pence, how many shillings will a hundredweight cost?

13. From a stick x inches long, p pieces, each of length y inches, are cut; how long is the piece that is left?

14. In a race A runs x yards a second, and B y yards a second; if B has a yards start, in how many seconds will A overtake B?

15. During the cricket season a boy scores x, y, 2x, 3y, 3x, 2y runs in successive innings. What is his average?

16. If $x = 1, y = 2, z = 3$, find the value of $2xy + 3yz - 4xy$.

17. When $x = 1, y = 2$, it is found that $2x + 3y - 4z$ equals 8; what is the value of z?

18. A man with x nephews gave them 5 shillings each and then had 7 shillings left in his pocket. How many shillings had he at first?

19. What is the meaning of $3x$? Find the value of
$$3x+4x-2x.$$

20. A man walked from a place P to a place Q and then to a place R. The distance from P to Q was x miles and it was three times as far from Q to R. He walked altogether 12 miles; how far was it from P to Q?

7. Example. *There are* x *boys in the 1st form, twice as many in the 2nd form, and four times as many in the 3rd form. How many are there in the three forms together?*

Number of boys $= x+2x+4x$
$= 7x.$

For just as 1 penny + 2 pennies + 4 pennies = 7 pennies
so $1x \quad +2x \quad +4x \quad = 7x.$

Terms which contain the same letter are called **like terms.**

Rule for adding like terms. Add up the coefficients of the several terms. The result is the coefficient in the answer, e. g. $3y+4y+6y = 13y$, since $3+4+6 = 13$.

Rule for adding any number of algebraical expressions. For convenience arrange the expressions one under the other, so that like terms come in the same column (just as £ *s. d.* are arranged in compound addition), then add up the several columns of like terms, and insert + signs between the several results.

The result of adding quantities is called their sum.

Example. *Add together*
$$2a+3b+4c,\ 3a+b+2c,\ a+2b+4c.$$
$$\begin{array}{r} 2a+3b+4c \\ 3a+\ b+2c \\ \ a+2b+4c \\ \hline 6a+6b+8c \end{array} \quad \text{Sum.}$$

Remember that the coefficient 1 is understood in such terms as $a, +b$, and must be added in with the other coefficients.

8. Example I. *A tradesman begins the day with £18; he sells goods to the value of £10 and is paid £12 owing to him; he loses £5, pays £8, and lends £4. How much has he at the end of the day?*

Amount at the end of the day $= 18+10+12-5-8-4$ pounds
$= 40-17$ pounds
$= £23$ *Answer.*

In this example the actual order of the various receipts and payments does not affect the result; we take no account of the order but add up all the receipts, add up all the payments and losses, and then subtract.

Example II. *A tradesman begins the day with £4y; he sells goods to the value of £3y and is paid £y owing to him; he loses £2y, pays £y, and lends £3y. How much has he at the end of the day?*

Exactly as in the arithmetical example we add up the quantities increasing his money, add up the quantities decreasing his money, and then subtract.

Amount at end of day $= 4y+3y+y-2y-y-3y$ pounds
$= 8y-6y$ pounds
$= 2y$ pounds.

Hence we deduce an extension of the addition rule.

Rule. If in a number of like terms some are to be added and some subtracted, first add all those that are to be added, then those that are to be subtracted, and subtract the second sum from the first.

The result is called the algebraic sum of the quantities.

Examples I d.

Add together:

1. $3x+4y-z$, $2x-y+2z$, $5x-2y+4z$.
2. $+2a-3b-4c$, $3a-2b+5c$, $4a+7b-c$.
3. $3a+2b-c$, $2a-b$, $-2c+3b+2a$, $a+b-c$.
4. $2p+q$, $5p+4q-r$, $4r-2q+3p$, $3q$.
5. $3x-4y+5z$, $-x+5y-2z$, $2x+2y+4z$.

Simplify:

6. $4x-2y+3z+4z-2x+3y-8x+y-4z+9x$.
7. $3a+4b-2a+9b+3a-c+2a-4b+6c+4a$.

8. $2ab+3bc-4ca-3ab+5ca-2bc+4ab.$
9. $3p-4x+2a-4p-3a+5x-2x+2p+3a+6x.$
10. $2b+2c-3a+4a+5b-c-a-4b+3c.$

If $a=3$, $b=2$, $c=1$, find the value of

11. $2ab+3bc-4ac.$
12. $3a\times 2b\div c+4b\times 3a\div 2c.$
13. $3a+4b-2a+2c-2b+3a-c.$
14. $c-2a+3b-4c+8b-3a.$
15. $4abc-2ab-3bc-ab.$

If $x=3$, $y=0$, $z=4$, find the value of

16. $3x+\frac{y}{z}-2z.$
17. $xy+4yz+3zx-4x+5y+6z-2xyz.$
18. $3x-4z+2y-x+2z-3y+17x-4z+3.$

9. Use of Brackets.

Example I. *A purse contains* 3 *compartments; each compartment contains one sovereign, one shilling, and one penny. What is the total value of the money in the purse?*

Method I. Value of money in each compartment = one sovereign + one shilling + one penny.

∴ value of money in purse = 3 times the result obtained by adding one sovereign, one shilling, and one penny.

Which we express thus:

value of money in purse = 3 (1 sovereign + 1 shilling + 1 penny).

This means that the quantities in the brackets ( ) are first to be added and the result is to be multiplied by 3.

Method II. There are altogether 3 sovereigns, 3 shillings, and 3 pennies.

i. e. value of money in purse = 3 sovereigns + 3 shillings + 3 pennies.

Comparing the two methods we see that

3 (1 sovereign + 1 shilling + 1 penny) = 3 sovereigns + 3 shillings + 3 pennies.

Example II. *On each day of the week, except Sunday, a man receives* 7 *shillings and pays away* 6*s.* 2*d. How much does he save per week?*

Method I. Amount saved per day $= 7s. - 6s.\ 2d.$
$\therefore$ amount saved per week $= 6\ (7s. - 6s.\ 2d.)$.

Method II. Total amount received $= 6 \times 7s.$
Total amount spent $= 6 \times 6s.\ 2d.$
$\therefore$ amount saved per week $= 6 \times 7s. - 6 \times 6s.\ 2d.$

Comparing the two methods we see that

$$6\ (7s. - 6s.\ 2d.) = 6 \times 7s. - 6 \times 6s.\ 2d.$$

Example III. *A traveller is at a height of* 10000 *feet; the next day he descends* 2000 *feet and ascends* 500 *feet. What is the height now?*

Method I. Since he first descends 2000 feet and then ascends 500 feet, he descends (2000 − 500) feet in the course of the day.
But he started the day at a height of 10000 feet.
$\therefore$ at the end of the day the height is 10000 − (2000 − 500) feet.

Method II. When he has descended 2000 feet the height is 10000 − 2000 feet.
He now ascends 500 feet.
$\therefore$ final height = 10000 − 2000 + 500 feet.

Hence we see that $10000 - (2000 - 500) = 10000 - 2000 + 500$.

Example IV. *A traveller is at a height of* 10000 *feet; on each of three successive days he descends* 2000 *feet and ascends* 500 *feet. What is the height at the end of those three days?*

Method I. During each day he descends (2000 − 500) feet.
$\therefore$ in three days he descends 3 (2000 − 500) feet.
$\therefore$ at the end of the three days the height is 10000 − 3 (2000 − 500) feet.

Method II. During the three days he descends 6000 feet, and ascends 1500 feet.
$\therefore$ at the end of the three days the height is 10000 − 6000 + 1500 feet.

Hence we see that $10000 - 3\ (2000 - 500) = 10000 - 6000 + 1500$.

There would be no difference in the working of the previous examples if we had used letters instead of numbers.

Examples I e.

Each of the following examples should be worked by the two methods: (i) using brackets, (ii) not using brackets.

1. A man earns x shillings a week and spends y shillings a week. How much has he saved at the end of a year?

2. A man begins the year with £100. He earns £10 a month and spends £a a month. How much has he at the end of the year?

3. If, in question 2, a is greater than 10, how much has he at the end of the year?

4. In a farmyard there are three sheds; in each shed there are x fowls and y ducks. What is the total number of birds?

5. On each of four days of a week a merchant gained £x and lost £y. On each of the remaining week-days he lost £x and gained £y. If x is greater than y, what were his gains for the whole week?

6. A gardener earns p shillings a week and his son q shillings a week; how many shillings will they together earn in 5 weeks?

7. There are 4 bookshelves in a room; each of them contains 10 English books, 8 German books, and 7 French books. How many books are there in the room?

8. A man has 5 children; each child has x sons and y daughters. How many grandchildren has the man?

10. We see from the preceding examples * that we may omit brackets if we are careful to observe the following rules:

(i) If there is a + sign only before the bracket, the quantities inside the bracket are written down unchanged.

$$\text{e.g. } +(a-2b+3c) = +a-2b+3c.$$

If no sign is written before the first term inside a bracket the sign + is understood.

(ii) If there is a factor preceding or following the bracket, that factor multiplies each of the terms inside the bracket;

* The student must be warned that a rule cannot be proved by considering a particular case; we are, however, led to see what the rules are by considering several special cases; in later chapters proofs of these rules will be given.

if the factor or bracket is preceded by the sign +, the signs of the products are the same as the signs of the corresponding terms inside the bracket.

$+7(x-2y) = +7x-14y$ and $+(x-3y)a = ax-3ay$.

(iii) If there is a − sign only before the bracket, the sign of each quantity inside the bracket must be changed when the brackets are omitted; all the + signs become −, all the − signs +; e. g. $-(2x+3y-4z) = -2x-3y+4z$.

(iv) If a factor precedes the bracket, and a − sign precedes that factor, then each term inside the bracket must be multiplied by the factor, but the sign of each product is opposite to the sign of the corresponding term in the brackets; e. g. $-3(4a-2b+5c) = -12a+6b-15c$.

The process of omitting brackets is usually called 'removing brackets'.

Examples I f.

Examples for Practice in removing Brackets.

Remove the brackets from the following expressions, and simplify the results as far as possible:

1. $2(2x+y-z)$.
2. $3a(2x-y+z)$.
3. $3x-2(x-y)$.
4. $6-3(5-4)$.
5. $7+2a(b-c)$.
6. $7x-3(x-y)+2(y+x)$.
7. $(x+y)+(x-y)$.
8. $(x+y)-(x-y)$.
9. $(3a-b)+5(b-a)$.
10. $3a-4(x-2y)$.

11. In an examination A gets m marks, B gets n marks, and C gets twice the difference between A's and B's marks. How many marks has A more than C?

12. A and B have each 30 shillings and C has 15 shillings. A gives x shillings to B and three times as many to C, while C gives y shillings to B. How many shillings has each then? How many have they altogether?

13. Add together $(3a-2b)x+(a-3b)$ and $(3a+2b)x-(a+3b)$.

14. Add $x+2(y-z)$, $y-2(z-x)$, $z-2(x+y)$.

15. Simplify $3(4+5-1)-2(2+3-4)+4(3+4-5)$.

16. Find the sum of $2a-(5c+3b)$ and $4b-(a-3c)$; and subtract $(a-3c)$ from the result.

17. If $a = 3$, $b = 5$, $c = 0$, and $d = 1$, find the value of

$$\frac{(4a-3c)d}{(3b-5d)} + \frac{5abc}{9(bc+ad)} - \frac{(2a-4d)}{3b+c}.$$

18. Add $2(a-3b-2c)$, $3(b-4c+2a)$, and $4(2c-a+5b)$.

Examples I.

Miscellaneous.

1. An article costs £a; what is its value in shillings? How many could be bought for x shillings?

2. From $7a+2b+3c$ take $3a+b+2c$.

3. Add together $2a+4b-2c$, $3a-5b+6c$, $2b-4a+c$.

4. Simplify $3(2a-b)+2(a+2b)-4(a-2b)$.

5. An army consists of x regiments, each regiment contains y companies of z men each. How many men are there in the army?

6. Three boys spend their money on cakes. The first has a pounds, the second b shillings, and the third c pence. If each cake costs x pence, how many cakes can they buy?

7. What is the difference between a term and a factor? Write down two expressions, one containing three terms the other containing three factors.

8. A boy is x years old in the year 1908; how old will he be in 1917? When will he be 21 years old?

9. What must be added to $2a+b+4c$ to make $8a+4b+5c$? Verify the answer by putting $a = 1$, $b = 2$, $c = 5$.

10. A farmer sold a fowls at b pence each, b lambs at a pounds each, and with the money so obtained bought $(a+b)$ geese at c shillings each. How much money had he left?

11. If $p = 0$, $q = 1$, $r = 3$, find the value of

$$p(q-r)+q(r-2p)+3r(p+4q).$$

12. To each of 10 old women there was given a present consisting of a lb. of tea and b lb. of sugar; what was the total weight given away?

13. Simplify $2(-x+y)-3(x-y)+4(3x+2y)$.

14. There are three consecutive numbers of which the middle one is x; what are the others?

15. A boy returned to school with 15*s.* in his pocket; an uncle gave him x shillings, his father five shillings, and his mother a florin. What is the value of x?

16. Multiply $3a+4b+5c$ by $2x$.

17. A boy walks a miles b furlongs on each of three days in the week, $2b$ miles a furlongs on each of three other days, and b miles $2a$ furlongs on the remaining day. How many furlongs did he walk that week?

18. Find the sum of $2x+3y-z$, $3x-4y+6z$, $-x-y+5z$, and $-2x+6y-7z$.

19. If $a=2$, $b=3$, $c=0$, find the value of

$$3a+4b-2a(3b-4c)+3c(14a-2b).$$

20. A boy is x years old in 1908; in what year was he born? When will he be $2x$ years old?

21. Think of a number, double it, add 5, subtract 3, divide by 2, take away the number thought of and the remainder is always 1. Prove this to be true by taking x as the number.

22. When $a=3$, $b=5$, $c=7$, prove that

$$a(b+2c)+3b(c+2a)+3c(a+2b)=7ab+5ac+9bc.$$

23. What number is it (call it x) which when 5 is added to it is 7 less than double the number?

24. What is the remainder when $4a+5b+8c$ is diminished by $a+b+c$?

25. How many times will a bicycle wheel turn round in going x miles if it turns round once in every y yards?

26. When two numbers are added their sum is 96; one of them is x, what is the other?

27. If x is an even number, what is the next odd number?

28. Find the value of

$$3+5(6-\tfrac{12}{4})-3(2+2.4)+12(0+\tfrac{2}{8}).$$

29. Find the value of $(2a+3b-4c)-(a+2b-5c)$.

30. Subtract $a+2b-5c$ from $2a+3b-4c$.

31. Remove the brackets and simplify

$$-5(a-b-c)+30(b-c-a)+6c-4(c+b-a).$$

32. A regiment originally contained x men; after y men had been killed, the regiment was divided into p companies. How many were there in each company?

33. If 19 cricket balls at x shillings each cost £4. 15*s*. what is the value of x?

34. If I have in my purse a pounds b shillings, and I spend c shillings and c pence, how many pence have I left?

35. Write down five consecutive numbers so that x shall be the middle number of the five. What is their sum?

36. Distinguish clearly between the meanings of the algebraical expressions $\frac{a+b}{c}$, $a+\frac{b}{c}$, $\frac{ab}{c}$; and give the value of each expression when $a=1$, $b=2$, $c=3$.

37. If a train travels a yards in b seconds, how long will it take to travel 1 mile?

38. If a train travels at the rate of a miles an hour, how many hours will it take to travel b miles and how many seconds will it take to travel b yards?

39. Add together $a+(2b+3c)$, $2a-(b+2c)$, $b-(c+a)$, and $c-(a+b)$.

40. What must be subtracted from $5y+2x+8$ so as to leave $3y+2x+7$?

Find the value of $\frac{n}{2}(a+l)$ in the following cases:

41. $n=17$, $a=5$, $l=17$.
42. $n=5$, $a=7$, $l=109$.
43. $n=16$, $a=17$, $l=95$.
44. $n=20$, $a=1$, $l=6$.

Find the value of $\frac{Prt}{100}$ when

45. $P=115$, $r=4$, $t=5$.
46. $P=725$, $r=6$, $t=2$.
47. $P=136$, $r=5$, $t=5$.

48. Explain the words term, factor, coefficient, quotient; and give examples.

49. How many shillings change will there be out of a sovereign after paying a bill for a pints of milk at b pence per quart and c dozen eggs at the rate of d shillings a dozen?

50. A man starts with a sovereigns b shillings and c pence in his purse. How much money has he in all, expressed in pounds? Supposing him to pay away d sovereigns and f pence and to receive e sixpences, how much money has he left, expressed in pence? How many coins had he at first?

51. A man's salary is £x a day and he spends £y a day. How much does he save in z days? How long will he take to save £100?

52. Simplify $(2a-\frac{3}{4}b+\frac{1}{3}c)+(-\frac{1}{2}a+2b-\frac{1}{2}c)-(\frac{3}{4}a-b+2c)$.

53. From $2b-(c-a)$ take $2c-(a-b)$.

54. x and y are two numbers; express algebraically the two facts that their sum is 620 and their product 3684.

55. If I buy a apples at p pence a dozen and b apples at q pence a dozen and sell them all at one penny each, how much profit do I make?

56. A brick is a inches long, b inches broad, c inches thick; what is the length of all the edges added together?

57. A has 6 more marbles than B; B has x marbles. If A were to give one-third of his marbles to B, how many would B then have?

58. Find the number of pence in a sum of money made up of a guineas, b half-sovereigns, and c half-crowns.

59. If $x = a + b$ and $y = a - b$, find the value of $3x + 4y - 2a + 3b$. Verify by putting $a = 2$, $b = 1$.

60. The prices of seats in the stalls, pit, and gallery of a theatre are respectively seven shillings and sixpence, half a crown, and one shilling. Find in sixpences the total value of x stall, y pit, and z gallery tickets. What is the value in pounds when $x = 40$, $y = 200$, $z = 300$?

CHAPTER II

THE USE OF SQUARED PAPER

11. It is often possible to illustrate an arithmetical or algebraical argument by means of squared paper. This is done by taking a side of a square to represent the unit

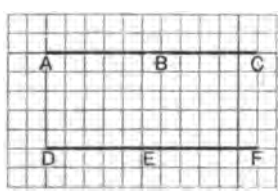

Fig. 1. Scale: side of small square represents 1 unit.

number; or it may be convenient to take some multiple or fraction of the side of a square to represent the unit. Every diagram should show clearly what scale is used.

Example I. *To show that* $6+5=5+6$.

Take a line AB to represent 6.

Produce it to C so that BC represents 5.

Then AC represents $6+5$.

Again, take DE to represent 5; produce it to F so that EF represents 6.

Then DF represents $5+6$.

It is seen that AC and DF are exactly the same length.

Hence the figure shows that $6+5=5+6$.

In a similar way the process of adding may always be represented, however many numbers have to be added.

Example II. *To show that* $17-9=8$.

Take AB to represent 17 units.

Measure BC from B towards A to represent 9 units.

Then from 17 units we have now taken 9 units.

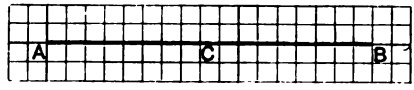

FIG. 2. Scale: side of small square represents 1 unit.

Hence AC represents $17-9$ units.

But by counting squares we see that AC represents 8 units.

Therefore $17-9=8$.

Example III. *To show that* $4\times 3=12$.

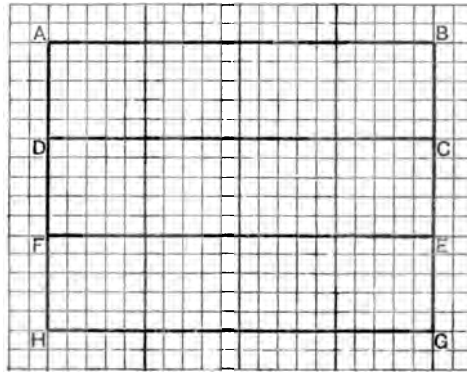

FIG. 3. Scale: 1 large square represents 1 unit.

It is sometimes found convenient to take the square itself to represent the unit.

In Fig. 3 1 large square represents 1 unit.

$\therefore$ 4 large squares represent 4.

i. e. the area $ABCD$ represents 4.

So also do the areas $DCEF$, $FEGH$.

$\therefore$ the whole area $ABGH$ represents $4+4+4$.

But 4×3 means $4+4+4$.

$\therefore$ area $ABGH$ represents 4×3 units.

But the area $ABGH$ is found by counting to contain 12 large squares.

Therefore Fig. 3 shows that $4 \times 3 = 12$.

Note. Any area such as $ABGH$ is called a rectangle.

Example IV. *To show that the number of units of area in a rectangle is equal to the number of units of length in one side multiplied by the number of units of length in the adjacent side.*

We now have two different kinds of unit, the unit of length and the unit of area.

Take the side of one big square to represent the unit of length.

Then, in Fig. 3, the side AB is 4 units long and the side AH is 3 units long, so that the number of units of length in one side multiplied by the number of units in the adjacent side $= 4 \times 3 = 12$.

But if we take the square whose side is unit length to be the unit area, that is, if we take 1 large square to represent unit area, the rectangle is seen to contain 12 units of area.

Hence Fig. 3 shows, for a particular case, that the number of units of area in the rectangle is equal to the number of units of length in one side multiplied by the number of units of length in the adjacent side, provided the unit of area is the square whose side is unit length. The same argument could be applied in all cases when the sides of the rectangle contain an exact number of units.

Example V. *To show that* $x \times y = y \times x$ *when* x *and* y *are whole numbers.*

Draw a rectangle $ABCD$ having AB x units long, AD y units long.

Then, as is shown in Ex. 4, the area of the rectangle is xy units of area.

Now suppose the rectangle turned round so that BC is in same direction as AB was before.

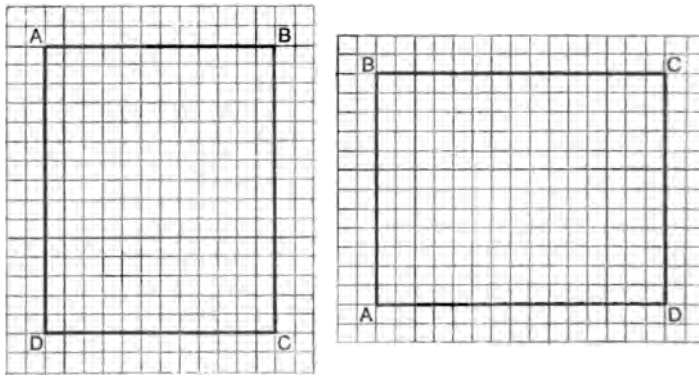

FIG. 4.

Then the area is now yx.

But since we are dealing with the same rectangle the area is the same.

Therefore $xy = yx$.

12. Powers and Indices.

Draw a square with side x units.

Then since both the adjacent sides are x units long, the area is xx units of area. Hence the product of two equal factors gives the area of a square; for this reason xx is called **the square of x** or **x square.**

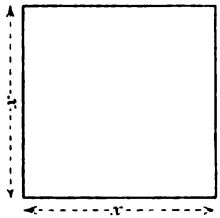

FIG. 5.

Similarly, by means of cubes, each having the edges 1 unit long, it can be shown that the volume of a cube whose edge is x units long is xxx times the cube

with edge 1 unit. If the volume of the smaller cube is taken as the unit of volume, the volume of a cube whose edge is x units of length is xxx units of volume. Hence the product xxx is called the **cube of x** or **x cube.**

But we may have any number of equal factors multiplied together. If we start with 1, five successive multiplications by x give $xxxxx$.

The result of multiplying 1 by a number of equal factors is called a Power of that factor. If there are five equal factors, the product is called the fifth power, i. e. if one is multiplied five times by x the product is the fifth power. Instead of writing the product in full, we write the factor only once, but put a small figure at the top right-hand corner to show the number of equal factors, e. g. instead of $xxxxx$ we write x^5. This little figure is called an **Index.** The index 1, like coefficient 1, is always omitted. x is called the first power of x, x^2 means xx, and, as stated above, is called x square, x^3 is x cube, x^4 is read 'x to the fourth', and, similarly, **x^n means the product obtained by multiplying 1 by x n times, and is called x to the nth.**

13. An expression may contain different powers of the same letter, e. g. $3x^3+4x^2-2x+6$. This means that when we know the value of x, we must take three times the cube of that value, add four times the square, take away twice the first power of the value, and finally add 6.

When adding or subtracting, different powers are treated in the same way as different letters.

Example I. *Add together* $3x^3-4x^2+2x-1$, $2x^2-3x+4$, x^3+3x^2+4x-2.

$$\begin{aligned}\text{Sum} &= (3x^3-4x^2+2x-1)+(2x^2-3x+4)+(x^3+3x^2+4x-2)\\ &= 3x^3-4x^2+2x-1+2x^2-3x+4+x^3+3x^2+4x-2\\ &= 3x^3+x^3+3x^2+2x^2-4x^2+2x+4x-3x+4-1-2\\ &= 4x^3+x^2+3x+1.\end{aligned}$$

Or the work may be arranged in columns:

$$\begin{array}{r} 3x^3-4x^2+2x-1 \\ 2x^2-3x+4 \\ x^3+3x^2+4x-2 \\ \hline 4x^3+\ x^2+3x+1 \quad \text{Sum.} \end{array}$$

Examples II.

Squared paper is to be used for the Examples 1–10.

1. Show that $9+6-7=9-7+6$.
2. Show that $3(a+b)=3a+3b$.
3. Prove that $a+b=b+a$.
4. Try to find a meaning for $6-9$.
5. Show that $a(a+b)=a^2+ab$.
6. What does $(a+b)^2$ mean? Show that $(a+b)^2=a^2+2ab+b^2$. Verify this when $a=4$, $b=3$.
7. What is the value of $2-4+3$?
   What is the value of $4-2-3$?
8. Taking the side of a square to represent a mile, represent in a diagram the following places:

(i) A at the intersection of two lines; (ii) B 3 miles east of A and 2 miles north; (iii) C 4 miles east of A and 3 miles south; (iv) D 5 miles north of C and 6 miles west; (v) E 2 miles south of D and 3 miles west.

How far is A from C?

9. Show that $4(a-b)=4a-4b$.
10. Show that $(a+b)+(a-b)=2a$ and $(a+b)-(a-b)=2b$.

Find the value of each of the expressions from 11 to 15, when $a=1$, $a=0$, and $a=4$.

11. $3a^3-2a^2+2a+4$.
12. $a^4+2a^3-4a+18$.
13. $3a^3+2a^2-3a-2$.
14. a^5-4a^4+3a+2.
15. $2a-4a^2+3-2a^3+5a^2+a^4$.
16. Find the sum of $3x^2-4x+6$, $3x-2x^2+4$, $6x-4$.
17. What must be added to nothing to make $3x^3-x^2+1$?
18. From $4a^2+5ab+b^2$ take $3a^2+2ab$.
19. What is the sum of

$$2x^3+4x+5,\ 3x^2-x^3-2,\ 4+x+3x^3-x^2-3x+1\ ?$$

20. Simplify $x(x-2)-3(x^2-1)+4x(1+x)$.

21. Multiply x^2 by x^3.

22. Find the value of $3abc-2ab+4bc-3ac+2x^2$ when each of the letters equals x.

23. If $a = 2$, $b = 3$, find the value of

(i) a^2+ab+b^2; (iii) $3(2a+b)-2(2a-b)$;
(ii) $3a^2-4ab+2b^2$; (iv) $a^2b+2ab^2-3a^3+2b^3$.

24. Find the value of $y-(x-x^2-1)$ when $y = 0$.

25. What must be taken from nothing to leave x^2-x+1?

26. A box is $2x$ inches long and $3y$ inches wide; what is the area of the top?

27. Find the area of a carpet whose length is x feet and breadth x inches.

28. A garden is x yards long and y yards wide. What is the cost of covering it with gravel at a shillings per square yard?

29. On a tour a man spent every day as many pence per mile as he walked miles. He walked x miles the first day, $2x$ the second day, and $3x$ the third day. How many shillings did he spend?

30. During a certain lesson each boy in a row obtained as many marks as there were boys in the row. There were 8 boys in the front row, 7 in the second row, and 4 in the third. How many marks were obtained altogether?

31. A square grass plot is surrounded by a gravel path 4 feet wide. If the side of the square is x feet, what is the area of the path? Illustrate your answer with a diagram, taking $x = 20$.

32. How many cubic inches are there in a brick x inches long, y inches wide, y inches thick? How many such bricks are required to build a wall a yards long, b feet high, and c inches thick?

33. During each lesson on a certain day a master gave each boy for each answer as many marks as he had answers. If a boy answered twice in the first lesson, x times in the second lesson, and $2x$ times in the third lesson, how many marks had he?

CHAPTER III

NEGATIVE QUANTITIES

14. Example. *A boy has in his pocket* x *marbles, and* y *of them are lost through a hole in the pocket. How many has he left? What does the answer become when* (i) $x = 20$, $y = 4$; (ii) $x = 20$, $y = 20$; (iii) $x = 20$, $y = 25$?

Number of marbles left $= x - y$.

In case i, number left $= 20 - 4 = 16$.

In case ii, number left $= 20 - 20 = 0$.

In case iii, number left $= 20 - 25$.

Now it is impossible to take 25 from 20, so that we cannot find an answer in case iii.

Returning to the question, we see that if a boy has only 20 marbles it is impossible for 25 to drop through a hole in the pocket. Hence in this example we are not surprised to find that we cannot answer the question.

But some examples can be worked out in two different ways, and by comparing the two answers so obtained we can very often find a meaning for the result when a number is subtracted from a number smaller than itself.

15. Example. *A boy has* x *shillings in his pocket, but owes* y *shillings. What has he left after paying his debts? What does the answer become when* (i) $x = 5$, $y = 3$; (ii) $x = 5$, $y = 5$; (iii) $x = 5$, $y = 7$?

After paying debts he has left x shillings $- y$ shillings $= (x - y)$ shillings.

$\therefore$ in case i he has left $(5 - 3)$ shillings $= 2$ shillings.

in case ii he has left $(5 - 5)$ shillings $= 0$ shillings.

i.e. he has nothing left.

in case iii he has left $(5 - 7)$ shillings.

We might try to take 7 away in two steps, first take 5 away and then 2. This would give $5 - 7 = 5 - 5 - 2 = 0 - 2$.

So that the number of shillings left is the number left when 2 is taken from 0, and this cannot be done.

0−2 is two less than nothing; the zero is usually omitted, and we are left with the symbol −2 to represent two less than nothing.

But in case iii we really have the following question:—
A boy has 5 shillings in his pocket, but owes 7 shillings. What is left after using his money to pay his debts?

He still owes (7−5) shillings.

i. e. he still has a debt of two shillings.

Comparing these two results, we see that having −2 shillings is the same thing as having a debt of 2 shillings.

Examples III a.

In the following set of examples each result should be worked for the three cases: (i) $x = 5$, $y = 3$; (ii) $x = 5$, $y = 5$; (iii) $x = 5$, $y = 7$. Try to find a meaning for all the answers.

1. A man stands with his back against a wall; he walks x yards forwards and then y yards back. How far is he now from the wall?

2. In the course of a certain day a merchant gains £x on some transactions, but loses £y on others. What is his gain on the day's work?

3. A man walks x miles due east from his house; he turns round and walks y miles in the opposite direction. How many miles east of his house is he now?

4. A labourer goes up x rungs of a ladder and comes down y rungs. How many rungs above the ground is he now?

5. Suppose that in example 4 the ladder rested in a hole with the tenth rung level with the ground; he now goes up x rungs from the ground and comes down y rungs. How many rungs above the ground is he now? How many rungs above the bottom of the hole is he?

Illustrate by means of a diagram on squared paper.

6. A cyclist starts from the twentieth milestone on the road from London to Brighton. He cycles x miles towards Brighton and then y miles towards London. How far is he now from his starting-point? How far is he from London?

Illustrate by means of a diagram on squared paper.

7. In the Centigrade thermometer the freezing-point of water is 0° C. At 8 a.m. the reading is 0° C.; the thermometer rises x degrees, then falls y degrees. What is now the temperature?

8. A shopkeeper starts the day with £10 in his till; in the course of the day he takes £x from customers and pays bills to the amount of £y. What is his gain during the day? How much has he at the end of the day?

16. The student will have found that in several of the above examples a clear meaning can be attached to results which involve subtracting a number from nothing, and should now have little difficulty in understanding the following statements:

If we start with nothing and continually add on one, we obtain the ordinary whole numbers used in Arithmetic. These are called **Positive** numbers, and are sometimes distinguished by the + sign placed before them. As a rule the sign is omitted.

If we start with nothing and continually subtract one, we obtain another series of numbers which are called **Negative** numbers. These are distinguished by the sign -1; thus -5 denotes the negative number 5. The negative sign is never omitted.

17. Meaning of + and −. We now have two meanings for these signs. They may be used as verbs, in which case they give directions to add or subtract respectively; with this meaning they are **signs of operation.** They may also be used as adjectives, in which case they show whether the following number is positive or negative; with this meaning they are **signs of affection.**

When negative numbers are preceded by a symbol of operation they are enclosed in brackets; so too are positive numbers if the positive sign is inserted. Thus $5\times(-6)+(-3)\div(+2)-(-6)$ means multiply together the positive number 5 and the negative number 6, add the result obtained by dividing the negative number 8 by the positive number 2, and then subtract the negative number 6. In the next chapter we shall show how these operations are peformed.

18. There is another way of explaining negative numbers.

Suppose we start counting, and imagine the numbers arranged in a long straight line:

0. 1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12.

If we wish to add 3 to 8, we start at 8 and then count forward three places, so obtaining 11.

If we wish to subtract 3 from 8, we start at 8 and then count three places backwards, so obtaining 5.

If we start with zero and count forward any number of places, say five, then we come to the number 5.

But what do we obtain if we start with nothing and count in the reverse direction? It has just been seen that counting three places backwards gives the same result as subtracting three; therefore counting three backwards from nothing will bring us to the number $0-3$, i. e. -3.

Hence we may say that Positive Numbers are obtained by counting forwards from nothing, and Negative Numbers are obtained by counting in the reverse direction.

Consider a lot of letters in a row:

				1	2	3	4	5	6
A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
−3	−2	−1	0						

Start counting at any letter and then continue counting. In the above line we have started at *E*, so that *J* is numbered 6. As we go to the right the numbers increase, but if we start at *J* and go to the left the numbers decrease one at a time; hence *D* should be numbered one less than 1, i. e. nothing or zero, *C* one less than zero, i. e. -1, and similarly $B-2$, $A-3$.

Graphic Representation of Negative Numbers.

19. By means of squared paper we can very simply obtain a geometrical representation of both positive and negative numbers. Such a representation is called a **Graphic representation.**

Take any point on one of the lines and call it *O*.

Then, as has been shown in § 11, any positive number, 9 for instance, may be represented by drawing a line from

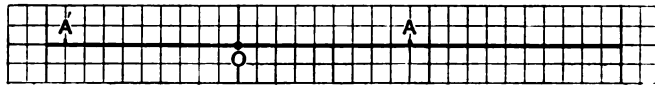

FIG. 6. Scale: side of 1 small square = 1 unit.

O to the right[1] equal to nine times the unit length. To do this we have to count nine squares to the right. The question arises, 'What will the line represent if we draw it to the left?' To do this we shall have to count in the reverse direction; but the last paragraph showed that by counting in the reverse direction we obtain negative numbers. Hence a line drawn to the left represents a negative number. In Fig. 6 OA represents $+9$, but OA' represents -9. Similarly if a line drawn upwards[1] represents a positive number, a line drawn downwards will represent a negative number, e.g. in Fig. 7 $OB = +9$, $OB' = -7$.

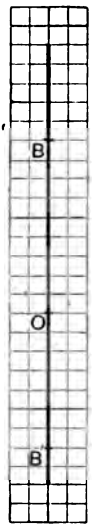

FIG. 7.

20. There is yet another way of regarding negative numbers. It has been mentioned in § 16 that any positive number can be obtained by continually adding one to nothing, i.e. $+5 = 0+1+1+1+1+1$; and that any negative number can be obtained by continually subtracting one from zero, i.e. $-5 = 0-1-1-1-1-1$.

Hence $+5+(-5) = 0+1+1+1+1+1$
$+0-1-1-1-1-1 = 0.$

[1] It is the *custom* to represent positive numbers by lines drawn towards the right or upwards; in such cases a line drawn to the left or downwards will, as is here shown, represent a negative number. If in any particular case it is more convenient to draw the positive lines to the left or downwards then lines drawn to the right or upwards will be negative.

Or $+x+(-x) = 0+1+1+1 \ldots$ the addition being repeated x times
$+0-1-1-1 \ldots$ the subtraction being repeated x times
$= 0.$

In words, this last result shows that any negative number when added to the positive number of the same size exactly cancels it, and the result is nothing. Thus a negative quantity is the exact opposite or reverse of the corresponding positive quantity, e. g. if $+6$ represents a distance measured to the right, -6 will represent an equal distance to the left.

Examples III.

1. If the rise in temperature during a certain period is represented by $x°$, what is meant when $x = -5$?

2. What is the result when the positive number 5 is added to the negative number 6?

3. By means of squared paper show that

$$5+(-6)+4 = 5+4+(-6).$$

4. Find the value of $3x$ when $x = 0$, when $x = +2$, and when $x = -3$.

5. A person A begins playing a game with x counters; he loses y, gains z, and then loses $2x$ counters. How many has he now? Find the value of the answer in each of the following cases:

(i) $x = 24,\ y = 6,\ z = 30$;
(ii) $x = 20,\ y = 3,\ z = 35$;
(iii) $x = 30,\ y = 10,\ z = 5$.

6. Find by squared paper or otherwise the value of $x+y$ when (i) $x = 0, y = -2$; (ii) $x = 3, y = -3$; (iii) $x = 1\frac{1}{2}, y = 2\frac{1}{2}$; (iv) $x = 1\frac{1}{2}, y = -2\frac{1}{2}$.

7. A boy possesses two shillings, but has been fined one shilling for breaking a window.

(i) If he is excused the fine, how much better off is he?

(ii) Some one gives him a shilling; how much better off is he?

8. What is left when y is subtracted from x? What must be added to y to make x?

Verify the results in the following cases: (i) $x = 5, y = 3$; (ii) $x = 5, y = -3$; (iii) $x = 5, y = 6$.

9. In the year 1908 a boy is x years old. In how many years will he be y years old? Verify by taking $x = 13$, $y = 17$ and $x = 13$, $y = 7$. Give the year in each case.

10. Find the value of xy in each of the following cases: (i) $x = +2$, $y = +3$; (ii) $x = -2$, $y = +3$; (iii) $x = +2$, $y = -3$; (iv) $x = -2$, $y = -3$.

11. What is the result of adding x to nothing nine times? What is the result of subtracting x from nothing nine times?

Explain the meaning of the negative quantity in each of the following cases:

12. A man gains -5 pounds.

13. An event happened -3 years ago.

14. The temperature fell -3 degrees.

15. A grocer lost -5 shillings every day.

16. The boy went up -5 places.

17. A man threw a stone up into the air a distance of -50 feet.

18. The match was lost by -5 runs.

19. The natives have -4 fingers on each hand.

CHAPTER IV

FURTHER CONSIDERATION OF THE SIMPLE RULES

Now that we have introduced negative numbers it is necessary to consider carefully what is meant by the simple operations Addition, Subtraction, Multiplication, and Division; and to find out whether it is possible to perform these operations with negative numbers.

Addition.

21. Consider the question in § 1, and suppose a child who could not add, but could count, were told to find out how many cows there were. He would go to the first field and

count up to 13; then to the second field and go on counting, the first cow in the second field being reckoned as 14, the next as 15, and so on up to 29. Similarly he would start at 30 in the third field, and finally find the total to be 34.

A little consideration will show that all simple addition sums in arithmetic can be done by counting, and in fact most beginners do addition sums by counting their fingers or strokes made on paper. Hence we see that addition is a short process to avoid a lot of counting.

It has been shown in § 18 that when we have a negative number we must count in the opposite direction to the direction we count when we have a positive number.

Suppose now we wish to find the value of $x+y$ in the following cases: (i) $x=+4, y=+3$; (ii) $x=+4, y=-3$; (iii) $x=-4, y=-3$; (iv) $x=-4, y=+3$.

Imagine the numbers arranged in a straight line as follows:

$-7.\ -6.\ -5.\ -4.\ -3.\ -2.\ -1.\ \ 0.\ 1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.$

Case i. $x=+4,\ y=+3$.

Starting from 0 we count to the right to 4, then go on counting three more numbers, so obtaining 7 as the answer.

Case ii. $x=+4,\ y=-3$.

Starting from 0 we count to the right to 4, then count backwards three numbers, so obtaining 1 as the answer.

Case iii. $x=-4,\ y=-3$.

Here we start from 0 but count to the left to -4, then go on counting in the same direction, so obtaining -7 as the answer.

Case iv. $x=-4,\ y=+3$.

Again we start from 0 counting to the left to -4, but then count backwards to the right, so obtaining -1 as the answer.

We have now shown that

$$+4+(+3)=+7$$
$$+4+(-3)=+1$$
$$-4+(-3)=-7$$
$$-4+(+3)=-1.$$

The student should go through the same series of additions with other numbers, as for example (i) $x=5,\ y=8$; (ii) $x=3,\ y=9$.

22. We now see that when a negative quantity is added the result is less than the original quantity, and that adding a negative quantity is the same thing as subtracting the corresponding positive quantity.*

Quantities which are all positive or all negative are said to be of **like sign**, quantities of which some are positive and some negative are said to be of **unlike sign.**

The general rules for addition are:

(1) If the quantities are of like sign add up the numbers and prefix + if the quantities are all positive, — if the quantities are all negative.

(2) If two quantities are of unlike sign, subtract that which is numerically smaller from the other, and prefix the sign of the greater.

Note.—Of the two numbers -8 and $+6$, -8 is really less than $+6$, but is said to be **numerically greater** than $+6$ because 8 is greater than 6.

(3) If there are any number of quantities, first add the positive quantities, then the negative quantities, and then obtain the final answer by Rule (2), e. g.

$$\begin{aligned} 5+(-2)+(+3)+(-7)+(+7) &= 5+3+7+(-2)+(-7) \\ &= +15+(-9) \\ &= +6. \end{aligned}$$

It should be noted here that we have assumed that additions may be performed in any order. This is obvious in the case of positive numbers (see § 11), and will be shown later to be true for negative numbers.

23. The same rules apply if, instead of mere numbers, we have quantities involving letters.

Example. *Find the sum of* $2a-3b+4c$, $3a+4b-2c$, $-5a+b-3c$.

Arrange the quantities so that the same letters come in the same vertical column.

* See note, page 20.

$$\begin{array}{r} 2a-3b+4c \\ 3a+4b-2c \\ -5a+\ b-3c \\ \hline +2b-\ c \end{array}$$ *Answer.*

[The mental work deals only with the coefficients.

$2+3=+5$ and -5 makes 0, $\therefore$ there are no a's.

$4+1=+5$ and -3 makes $+2$, put down $+2b$.

$-2+(-3)=-5$ and $+4$ makes -1, put down $-c$.]

The work could also be arranged as follows:

$$\begin{aligned} \text{Sum} &= 2a-3b+4c+3a+4b-2c-5a+b-3c \\ &= 2a+3a-5a-3b+4b+b+4c-2c-3c \\ &= 5a-5a-3b+5b+4c-5c \\ &= 2b-c. \end{aligned}$$

Subtraction.

24. Subtraction is the reverse operation to Addition, that is to say, it undoes the result of addition, e. g. if we add 7 to 8 we obtain 15; if we wish to reverse this we must subtract 7 from 15. Hence we expect that when we subtract we must do exactly the opposite to what we do when we add.

It has been shown in § 22 that in order to add a negative quantity we actually subtract the corresponding positive quantity; we therefore expect that in order to subtract a negative quantity we shall have to add the corresponding positive quantity. Let us see if this is reasonable.

Example I. *A boy has two shillings, and owes sixpence. How much has he really got? If he is excused the debt, how much has he?*

In the first case he really has $2s.-6d.=1s.\ 6d.$

In the second case he has $2s.$

So that taking away the debt has made him better off.

In symbols $1s.\ 6d.-(-6d.)=2s.=1s.\ 6d.+6d.$

Hence in this case subtracting a negative quantity has the same effect as adding the corresponding positive quantity.

Example II. *A boy owes two other boys* 6*d.* *and* 9*d.* *respectively. How much has he? If he is excused the larger debt, how much has he?*

It has been shown that having a debt is represented by having a negative number of pence (or pounds or shillings).

Therefore in the first place the boy has $-6d.+(-9d.) = -15d.$

In the second place the boy has $-6d.$

That is $-15d.-(-9d.) = -6d. = -15d.+9d.$

Hence in this case subtracting a negative quantity has the same effect as adding the corresponding positive quantity.

In working the following examples it is useful to recollect that there are two ways of regarding subtraction. 15—8 may be taken to mean 'What is left when 8 is taken from 15?' This is the ordinary subtraction idea; or it may be taken to mean 'What must be added to 8 to make 15?' From this point of view subtraction is sometimes called **complementary addition.**

Examples IV a.

1. What must be added to -3 to make -7?
   What is the value of $x-y$ when $x = -7$, $y = -3$?

2. A man had £x *y*s. *z*d. in his pocket; he lost £3 *z*s. *y*d. How much was left? Express the answer in pence. Try to work this by ordinary compound subtraction, and so verify your answer.

3. At the beginning of the year 1907 a merchant's property was worth £3514; at the end of the year he owed £56. How much had he lost during the year?

4. At the beginning of the year a merchant had £x; during the year he gained £y. How much had he at the end of the year?
   Explain the question and the answer in the following cases:
   (i) $x =$ £500, $y =$ £-700;
   (ii) $x =$ £-700, $y =$ £500.

5. At noon on Sunday the temperature was 20° F., at noon on Monday it was x° F. What was the rise in temperature? What was the fall? What do the answers become when $x = 29$?

6. During the day the thermometer rises from x° C. to y° C. What is the rise in temperature? What does this become if $x = -5$, $y = 7$?

7. Prove in any way that $-6-(-5) = -1$.

8. In a 100 yards race between A and B, A had x yards start and was beaten by y yards. How far had A run when B was at the winning-post? Explain the question and answer when

$$x = -3,\ y = -5.$$

25. The student will now see that the suggestion of § 24 is correct, and we can state the rule for subtraction in the following form:

To subtract any quantity from another, change its sign and follow the rules for addition given in § 22, thus:

$$3x-(-2x) = 3x+2x = 5x.$$

This rule applies for single quantities or for quantities consisting of several terms.

Example. *From* $3a-4b+5c$ *take* $2a+2b-4c$.

We can adopt either of the two following modes of setting down the work.

$$\begin{array}{r} 3a-4b+5c \\ 2a+2b-4c \\ \hline a-6b+9c \quad \textit{Answer.} \end{array}$$

[Mental work. $3a-2a=a$, $-4b-2b=-6b$, $5c+4c=9c$; the lower sign being changed in each case.]

Or Difference $= 3a-4b+5c-[2a+2b-4c]$

$$= 3a-4b+5c-2a-2b+4c$$
$$= 3a-2a-4b-2b+5c+4c$$
$$= a-6b+9c.$$

Notice that this agrees with what has already been shown in § 10 when dealing with the removal of brackets.

Graphical Illustration of Addition and Subtraction.

26. It has been shown previously that positive quantities may be represented by lines drawn to the right or upwards, and negative quantities by lines drawn to the left or downwards respectively.

To represent $+5+(-3)$ (Fig. 8).

Draw $OA = 5$ units to the right.

From A draw $AB = 3$ units to the left.

Then OB represents $5+(-3)$.

But $OB = +2$.

$\therefore\ +5+(-3) = +2.$

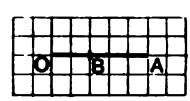

FIG. 8. Scale: side of 1 small square represents 1 unit.

To represent x+(−y) *when* x *and* y *are both positive and* x *greater than* y (Fig. 9).

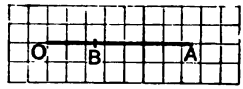

FIG. 9. Scale: side of 1 small square = 1 unit.

Draw OA to the right. As we do not know what x equals, we make OA any length; and as we are not told that x is a whole number, OA should not end at the end of a unit length.

Measure AB to the left; the length of AB is y. All that we know about y is that it is less than x, therefore B must be to the right of O.

Hence OB is positive and measures $x-y$ units.

$\therefore\ x+(-y)=+(x-y)$ when x and y are positive and x greater than y.

To represent x+(−y) *when* x *and* y *are both positive and* x *less than* y (Fig. 10).

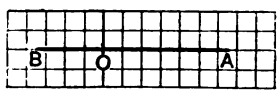

FIG. 10.

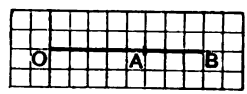

FIG. 11.

Scale: side of 1 small square = 1 unit.

Draw OA to the right to represent x units.

Measure AB to the left to represent y units. As y is greater than x, B is to the left of O, and $OB = y-x$ units in length, but is negative in sign.

$\therefore\ x+(-y)=-(y-x)$ when x and y are positive and x less than y.

In a similar manner the student can show that

$$(-x)+(-y)=-(x+y).$$

27. *To represent* +5−(−3) *graphically* (Fig. 11).

Draw $OA = 5$ units to the right.

If we were to add (−3) units, we should now draw $AB = 3$ units to the left; but as we are to subtract, we draw AB in the reverse direction, i. e. $AB = 3$ units to the right.

Hence $OB = 5+3 = 8$ units to the right.

$\therefore\ +5-(-3) = +5+3 = 8.$

To represent x—(—y) *graphically* (Fig. 12) *when* x *and* y *are positive.*

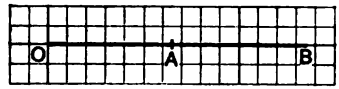

FIG. 12. Scale : side of 1 small square = 1 unit.

Draw OA to the right equal to x units.
Draw AB to the right equal to y units.
Then $OB = x+y$ units to the right.
$\therefore\ x-(-y) = x+y.$

Exercise. The student should show graphically that

$$-x-(-y) = -x+y,$$

and make two figures according as x is greater or less than y.

We have now shown graphically that the rules for addition and subtraction given in §§ 22 and 25 hold not only when the quantities are whole numbers but also when they are fractions.

Examples IV b.

1. Find the sum of $2a+3b-4c$, $3a-2b+c$, $-a+2b-c$, $2a-3b-5c$.
2. Subtract $3a-2b-c$ from $a+b+c$.
3. Simplify $3x^2-4x^3+7x-2+3x-6x+5-2x^3$.
4. If $a=5$, $b=3$, $c=2$, find the value of
   (i) $3a^2-2ab+bc-c^2+4bc-a^2$.
   (ii) $3a(b+c)-2b(4a-c)-c(b-c)$.
   (iii) $(a-b)+(b-c)+(c-a)$.
5. Add $3x^3-4x^2+1$, $3x^2+4x+x^3$, $3x+4+x^2$.
6. What must be added to $x+y+z$ to make $3x-4y-2z$? Verify by putting $x=1$, $y=z$, $z=0$.
7. Simplify $2ab-4a^2+2b^2-3ab+5a^2-2b^2+a^2-ab$.
8. From $2x^2-3x+1$ take $x+x^2-2$.
9. From $2a^2-3b^2+c^2$ take $3c^2+b^2-a^2$.
10. From $x^4-x^3+2x^2+3x-4$ take $2x^4+3x^3-2x^2+1$.
11. If a is greater than c, from £a bs. cd. take £c. bs. ad.
12. Find the sum of $4a^2-5ab+5ac-b^2-c^2$, $3a^2+2ab-c^2-2bc$, $-a^2-2ac+3b^2-5bc$, $b^2-2ac+c^2-3ab$.

13. From the sum of $3x^2-4x+1$ and $1-2x-3x^2$ take the sum of $2x^2-4x+3$ and $2x-1-x^2$.

14. Simplify $5+(-7)-(+5)+6-(-3)+4-(+8)$.

15. Add together $6a^3-(5a^2b-b^3)$, $7a^2b-(3ab^2-2b^3)$, $5ab^2-(4a^3+b^3)$, and subtract $2a^2b-2ab^2$ from the sum.

16. Add together $2a(a-2b-c)$, $b(4a-3b+5c)$, $c(2a-5b+c)$, and subtract a^2-4b^2 from the sum.

17. Find the sum of

$$3a^2-5a+2b,\quad 7a-4b-3b^2,\quad 2a^2-b+a-2b^2.$$

18. Find the value of $3x^2-5(x+7)-3(x^2-5x)-7$ when $x=4$.

19. When $a=2$, $b=0$, $c=-1$, find the value of

(i) $a(b+c)+b(c+a)$ and (ii) $(a+b-2c)^2-a^2+b^2-4c^2$.

20. Subtract $3x-y-2(x+y-z)$ from $3(x-y+z)$.

21. Add together $5a-(7b-c)$ and $3b-(9a+c)$, and subtract the sum from $c-4b$.

22. From $3x-(y-z)$ subtract $2z-(y-4x)$, and add $2x+y$ to the result.

23. If $a=2$, $b=3$, $c=5$, prove that a^2-ab+b^2 is greater than $bc+ca-c^2$.

24. What quantity must be subtracted from $-2x+3y-5z$ in order that the remainder may be $2x-7y-4z$?

25. Subtract the sum of $5a-\frac{4}{5}b+\frac{3}{5}c$ and $-\frac{3}{5}a+3b-\frac{1}{5}c$ from $\frac{4}{5}a+\frac{1}{5}b+c$.

26. To what expression must $10ab-22bc-14ac$ be added to produce zero? If $6x^2-14x+4$ be subtracted from zero, what will be the result?

27. What is the value of x when

(i) $x+2=6$; (ii) $x-2=-5$; (iii) $x+3=1$?

28. What number added to 16 will make -18? What must be added to -17 to make $+20$?

29. What must be subtracted from 16 to make 19?

Multiplication.

28. *Find the value of* xy *when* (i) $x=+4$, $y=+3$; (ii) $x=-4$, $y=+3$; (iii) $x=+4$, $y=-3$; (iv) $x=-4$, $y=-3$.

We will regard the second factor y as the multiplier and the first factor x as the number to be multiplied (the multiplicand, as it is called).

(i) $(+4)\times(+3)$ means $(+4)+(+4)+(+4)$ (§ 5)

$\therefore\ (+4)\times(+3) = 4+4+4$

$= 4\times 3$

$= 12.$

(ii) $(-4)\times(+3)$ means $(-4)+(-4)+(-4)$

$\therefore\ (-4)\times(+3) = -4-4-4$

$= -12$

$= -4\times 3.$

(iii) When the multiplier is negative, the interpretation of multiplication given in § 5 cannot be applied. We must therefore extend the meaning of multiplication, just as we have extended the meaning of addition and subtraction. Let us assume that when multiplying by a negative number we do exactly the reverse of what we do when we multiply by a positive number. Then instead of adding a certain number of times we shall subtract.

$(+4)\times(-3)$ will mean $-(+4)-(+4)-(+4)$

$\therefore\ (+4)\times(-3) = -4-4-4$

$= -12$

$= -4\times 3.$

(iv) $(-4)\times(-3)$ will mean $-(-4)-(-4)-(-4)$

$\therefore\ (-4)\times(-3) = 4+4+4$

$= +12$

$= +4\times 3.$

The argument would have been exactly the same if we had had any other whole numbers instead of 3 and 4.

Examples IV c.

1. Show that $(-5)\times(-6) = (-6)\times(-5)$.
2. When $a = 1$, $b = 2$, $c = 0$, $x = -1$, $y = -2$, find the value of
   (i) $2a+3b-5x$.
   (ii) $ab+bc+ca+xy$.
   (iii) $3ax-4by$.
   (iv) $a^2+b^2+c^2+x^2+y^2$.
   (v) $3a^2b-2ab^2+3xy^2$.
   (vi) $(a-b)\times(x-y)$.
   (vii) $2ax+2cy-3bx+4ay$.
   (viii) $3a^2x-2cy^2+3b^2x-4a^2y$.

(ix) $(a+b+c)-(x+y)$.

(x) $(ax+by)-x(ay-bx)$.

3. Show that $xy = yx$, whether x and y be positive or negative.

4. Find the value of $(2a+3b)(3x-4y)$ when $a=3$, $b=1$, $x=0$, $y=2$.

We can now state the rule.*

29. Rule for Multiplication. When two quantities are to be multiplied together, multiply the two numbers as in ordinary arithmetic, taking no notice of the signs, and then prefix + if the quantities are like (i. e. of same sign), — if the quantities are unlike (i. e. of different sign).

This is usually quoted in short as follows: 'Like signs give plus, unlike minus.'

It follows from this rule that a negative number multiplied by itself gives a positive product, so that the square of a negative number is positive, e. g.

$$(-5)^2 = +25, \quad (-\tfrac{1}{2})^2 = +\tfrac{1}{4}.$$

Division.

30. Just as in the case of subtraction, we may look at division in two ways. When dividing 12 by 3 we may regard ourselves as finding how many there would be in each part if 12 were divided into three equal parts, or as finding what number 3 must be multiplied by in order to give a product 12. This latter is the more general point of view, and gives us at once the rules for division.

We have $(+4)\times(+3) = 12$

$(+4)\times(-3) = -12$

$(-4)\times(-3) = +12$.

$\therefore +12 \div (+4) = +3$ [that is, $+4$ must be multiplied by $+3$ to give a product 12].

$(-12)\div(+4) = -3$

$(-12)\div(-3) = +4$

$(+12)\div(-3) = -4$.

The same argument applies to any numbers, and we have the rule.*

* See note, page 20.

Rule for Division. To divide one number by another, divide as in ordinary arithmetic, taking no notice of the signs, and prefix + if the quantities are like, — if they are unlike.

31. It has been shown that $a+b=b+a$, and $a\times b=b\times a$. Therefore we may talk of adding or multiplying two quantities. But $a-b\neq b-a$ * and $a\div b\neq b\div a$. Therefore **it is not correct to talk of subtracting two quantities or of dividing two quantities.** If we are to perform subtraction or division with two quantities, we must say '*subtract* a *from* b' or 'b *from* a', as the case may be; or '*divide* a *by* b' or 'b *by* a'.

Examples IV d.

1. Find the value of $xy+2xz-3x^2-2y^2+3yz$ when $x=2$, $y=-3$, $z=1$.

2. If $a=1$, $b=-2$, $c=3$, and $d=0$, find the numerical value of the expression $\dfrac{4a^2-b^2-2bc-cd}{3a-b(1-ac)}$.

3. If $a=1$, $b=-2$, $c=3$, and $d=0$, find the value of

$$\frac{b^2c^2}{a}+\frac{a^2c^2}{b}-\frac{b^2d^2}{c}.$$

4. When $a=6$, $b=5$, $c=-4$, find the numerical value of

$$2a(b^2-ac)-2b(b^2+ac)+\frac{2c-a}{a-b}.$$

5. Verify that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$

when $a=2$, $b=\frac{1}{2}$, and $c=-1$.

6. Find the value of $\dfrac{-2x^2+3x+1}{x^2-7x+2}$

(i) when $x=-3$; (ii) when $x=0$.

7. When $a=-3$, $b=-2$, $c=4$, find the value of $(a-b+c)^3$ and of $2a^3-3a^2b+4a^2c$.

8. If $a=0$, $b=1$, $c=2$, $d=3$, find the values of

(i) $a(b-c)+d(a-b)+c(d-a)$;

(ii) $(a-b)(b-2c)(c-3d)(d-4a)$.

* The sign $\neq$ means is not equal to.

9. If $a=3$, $b=5$, $c=0$, $d=-1$, find the numerical value of

$$\frac{(4a-3c)d}{3b-5d}+\frac{5abc}{9d}-\frac{2a-4d}{3b+c}.$$

10. If $a=1$, $b=3$, $c=-5$, and $d=0$, find the value of $5abc-8b^2c+15ac^2-14acd$.

11. When $a=-2$, $b=-3$, $c=4$, $d=6$, $e=0$, find the value of $a^2+b^2+c^2-d^2+e^2$ and of $3a^2-4ab^2+5cd^2-6abcde$.

12. What is the value of x when (i) $3x=-12$; (ii) $4x=16$; (iii) $-5x=20$; (iv) $-6x=-18$?

Examples IV.

1. Find the sum of $3x^3-2x^2+1$, $2x^2-x^3+x-4$, $5x^3-2x+3x^2$, $2+3x-x^2$.

Verify by putting $x=10$ and $x=-1$.

2. A man walked 5 miles east from his home, then x miles west, and then 2 miles east. He was then 3 miles from home. Find the value of x.

3. What is the meaning of x^3? Show that $x^5\times x^4=x^9$. Find as shortly as possible the value of 7^6 and 2^{10}.

4. If $x=0$, $y=1$, $z=-2$, find the value of

(i) $2x-3(2y-3z)$;

(ii) $2xy+\frac{4z}{y}-3yz^2$.

5. If x sheep cost £y, how many sheep will cost £z?

---

6. Find the value of

$$\frac{2ab}{cd}+\frac{3ac}{de}-\frac{4ad}{ce}+\frac{5(b+e)}{a+d}$$

when $a=-3$, $b=0$, $c=2$, $d=4$, and $e=6$.

7. A train runs a miles in x hours. Express its speed in feet per second.

8. Find the total cost (in shillings) of a drawing-boards at b pence each, a pencils at c shillings per gross, and a set squares at d pence per dozen.

9. State the rule of signs in Multiplication.

Multiply $4a-3b+5c$ by $-6x$.

10. A number x being taken, half of it is divided by 4 and the other half by 6. Express algebraically the sum of the two quotients; express also the difference between this sum and the quotient obtained by dividing the whole number x by 5. Supposing this difference to be 2, what was the given number?

---

11. Explain clearly the difference between 'index' and 'coefficient'. What are the rules for coefficients and indices in Multiplication (e. g. $3x^4 \times 4x^3$)?

12. Add together

$$-x^3-2ax^2+a^2x,\ 2x^3-ax^2,\ 4x^3-a^3,\ 5ax^2-a^2x-4a^3.$$

Test your answer by putting $x=2$, $a=-1$ in the four expressions and the sum.

13. If $a=1$, $b=-2$, $c=3$, $d=-4$, find the numerical value of

(i) $a^2+2ab-2acb^2-2bc+c^2$;

(ii) $\dfrac{a^2+b^2}{b+c}+\dfrac{b^2+c^2}{2a-d}+\dfrac{a^2+c^2}{b+d}$.

14. Multiply $3x^3-2x^2+x-1$ by $-2x^2$.

15. How many miles can a person walk in an hour if he walks a yards in 12 minutes?

---

16. To what expression must $3xy-22yz+13xz$ be added to produce zero? If $6p^2-11p-3$ be subtracted from zero, what will be the result?

17. If $a=0$, $b=1$, $c=-2$, and $d=4$, find the numerical value of

(i) $4bc-3b^2d+2abc+bcd$;

(ii) $\dfrac{b+c}{a^2+b^2}+\dfrac{2a+d}{b^2+c^2}+\dfrac{b+d}{a^2+c^2}$.

18. From $3ab+5cd-4abc-6bcd$ take $4cd+3ab-3abc-6bcd$.

19. B is x years old. Two years ago A was 5 times as old as B was 4 years ago. How old is A?

20. Is the following statement true when $x=3$:

$$14-x-21+11x-15=-5x-7+9x-14-3x+9?$$

---

21. If $y=3-2x$ and $z=3x-4y+2$, find the value of $3x-2y+3(z-2x)$ when $x=-2$.

22. What must $3a$ be multiplied by to give as the product $9a^2+6a-12a^3$?

23. A man gets x eggs for a certain sum of money, the price of each egg being y pence. How many would he get for the same sum if the price of each egg were $(y+z)$ pence?

24. Add together $3x^3-3ax^2+a^2x$, $-x^3+2ax^2-3a^2x+2a^3$, $3a^3-2x^3$, $2a^2x-3ax^2-2a^3$. Test the answer by putting $x=-1$, $a=-2$ in the four expressions and in the answer.

25. A boy had x shillings; he spent half his money on a bat, one-third on a ball, and had three shillings left. What was the value of x?

---

26. If $a=2$, $b=\frac{3}{2}$, $c=-5$, find the numerical value of $a^2+b^2-c^2-3ac+\frac{1}{4}ab+2bc$.

27. Distinguish between the meanings of (i) $(a+b)(a-b)$; (ii) $(a+b)a-b$; (iii) $a+b(a-b)$.

Find the sum of the three expressions.

28. How long will x men take to mow y acres of grass if each man mows z acres a day? How much longer would 3 fewer men take?

29. Is the following statement true when $y=-2$:

$$3y-2(y-6)+3=3(2-y)+5y+11\ ?$$

30. Prove by means of squared paper that

$$(x+3)(x-2)=x^2+x-6.$$

---

31. Find the value of

$$\frac{(-8)(-9)}{(-6)(-3)}+\frac{(-9)(+10)(-12)}{(-15)(+14)}-\frac{(-20)(+18)}{-(6+4)(-9)}.$$

32. Which quantities are to be multiplied together in the following expressions?

(i) $a+b\times c+d$; (iii) $(a+b)\times c+d$;

(ii) $a+b\times(c+d)$; (iv) $(a+b)\times(c+d)$.

Find the value of each expression when $a=2$, $b=3$, $c=4$, $d=5$.

33. Write down (i) the number of yards of carpet x feet wide required to cover a floor y feet long and z feet broad; (ii) a proper fraction whose numerator and denominator are consecutive odd numbers of which the larger is $2n+1$.

34. If $2x=1$, $3y=-1$, $5z=1$, find the value of

$$\frac{1}{x+y}+\frac{1}{x+z}+\frac{1}{y+z}.$$

35. A snail crawls up a pole 6 feet high. If the snail crawls up x inches every alternate minute, and slips down y inches in each of the other minutes, how long will it take to reach the top of the pole? Verify by putting $x = 3$, $y = 2$. (Assume that the snail takes an exact number of minutes.)

---

36. Add together $3(2x+4y+z)$, $-2(x+y+z)$, $2x-(4y-5z)$, and find the value of the sum when $x = y^2 = 100z$ and $z = 1$.

37. Find the sum of $4x^2y-(3xy^2+2y^3)$, $2x^3-(x^2y-3xy^2)$, $x^3+4x^2y-3y^3$.

38. Show that $x^9 \div x^3 = x^6$. Divide $x^8-2x^7+3x^6$ by x^6.

39. Show by using squared paper that $x(x+y) = x^2+xy$. Also draw a figure for the case when $x = 5$, $x = -3$.

40. A person who has c shillings buys a things each costing b pence; how many pence will he have left? Explain the answer if $c = 5$, $a = 9$, $b = 8$.

---

41. Find the value of $2c^2+2a^2+2b^2-4bc+6abcd$ when $a = 1$, $b = -2$, $c = -3$, $d = 0$.

42. From $5a^2b-6ab^2-7a^3b^3$ take $10a^2b+8ab^2-8a^3b^3$.

43. Show in any way that $(-2)\times(-3) = +6$.

Multiply $6a^3bc-7ab^2c^2$ by $-5a^2b^2$.

44. A man has x sovereigns and y shillings. Express in symbols the facts that he has 30 coins and that their value is £11.

45. How long will it take a train to go 100 miles if it goes x miles in y minutes?

---

46. When $a = 3$, $b = -2$, find the value of

$$\text{(i)}\ \frac{3a-4b}{6a+5b}; \quad \text{(ii)}\ \tfrac{1}{4}(4a-5b)(4a+5b).$$

47. Subtract $\frac{2}{3}a-5b-\frac{3}{4}c+\frac{1}{5}d$ from $3a-\frac{1}{3}b+\frac{1}{2}d-\frac{1}{4}c$.

48. What is the meaning of Division? Give correct answers to the following: (i) $x \div 3$; (ii) $£x \div 3$; (iii) $£x \div £3$.

49. Fill up the gap in the following statement:

$$7+(6-2)-(3-\ \ ) = 9.$$

50. A sum of two pounds is divided among 3 boys, A, B, and C, so that A has x shillings, B has 6 shillings more than A, and C 5 shillings less than A. Express this algebraically. What is the value of x?

CHAPTER V

BRACKETS

Before beginning this chapter the student should read once more §§ 9 and 10 in Chapter I. It is there shown how the need for the use of brackets arises, and how the brackets may be removed when no longer required. The following graphical illustrations of the use of brackets should be carefully studied:

32. *To prove that* $c(a+b) = ca+cb$ *when* a, b, c *are all positive.*

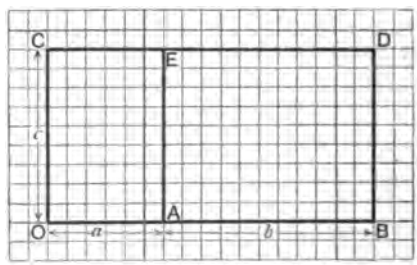

Fig. 13.

Draw OA equal to a units, and produce OA to B, making AB equal to b units.

Draw OC upwards equal to c units.

Complete the rectangle $OBDC$.

Now the length of OB is $a+b$.

$\therefore$ the area of the rectangle is $c(a+b)$. § 11. Ex. 4.

Mark off CE equal to a units, then $ED = b$ units.

Join AE.

Then area of rectangle $OAEC = ca$, and of rectangle $ABDE = cb$.

But the whole figure $OBDC$ = sum of $OAEC$ and $ABDE$.

$\therefore\ c(a+b) = ca+cb$.

Note. No scale is given in Fig. 13. Suppose now that the side of a square represents $\frac{1}{12}$ unit, then the area of a square is $\frac{1}{144}$ sq. unit.

Then $OA = \frac{6}{12} = \frac{1}{2}$ unit, $AB = 1\frac{1}{2}$ unit, $OC = \frac{9}{12} = \frac{3}{4}$ unit.

$\therefore$ area of whole rectangle $OBDC = \frac{3}{4}(\frac{1}{2}+1\frac{1}{2})$ square unit;

but area of $OAEC = 54$ squares $= \frac{54}{144}$ square unit,

area of $ABDE = \frac{90}{144}$ square unit.

$\therefore$ area of whole rectangle $OBDC = \frac{54}{144} + \frac{99}{144}$ square unit
$= \frac{3}{8} + \frac{33}{48}$ " "
$= \frac{3}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{11}{12}$ " "

Hence $\frac{3}{4}(\frac{1}{2} + \frac{11}{12}) = \frac{3}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{11}{12}$.

In a similar way it can be shown that $c(a+b) = ca+cb$ for any positive values, integral or fractional.

33. *To prove that* $c(a-b) = ca-cb$ *when* a, b, c *are all positive and* $a > b$.

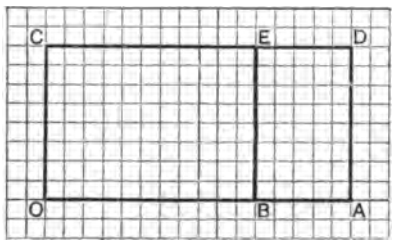

FIG. 14.

Draw a rectangle $OADC$ having $OA = a$ units, $OC = c$ units.

Measure AB backwards from A equal to b units and DE backwards from D equal to b units. Join BE.

Then OB equals $a-b$ units.

$\therefore$ area of rectangle $OBEC = c(a-b)$. § 11. Ex. 4.

Also area of rectangle $OADC = ca$ and of rectangle $BADE = cb$.

But rectangle $OBEC$ = rectangle $OADC$ − rectangle $BADE$.

$\therefore$ $c(a-b) = ca-cb$.

34. *To show that* $-c(a-b) = -ca+cb$ *when* a, b, c *are all positive and* $a > b$.

$-c(a-b)$ means subtract $c(a-b)$.

$\therefore$ $-c(a-b)$ means subtract $ca-cb$;

but in order to subtract we change the signs and add.

Hence $-c(a-b) = -ca+cb$.

35. *Find the sum of* $3x^2-2x+4$, ax^2+4x-c, $2x^2+bx+2$.

Arrange like powers in same column as directed in § 13.

$$\begin{array}{rrr} 3x^2 & -2x & +4 \\ ax^2 & +4x & -c \\ -2x^2 & +bx & +2 \\ \hline (1+a)x^2 & +(2+b)x & +(6-c) \end{array}$$

Answer.

[We have here followed the rule given in § 7. In the first column add up the coefficients of x^2, i. e. add $3+a-2$; this gives $1+a$. Therefore $(1+a)$ is the required coefficient of x^2. Hence we have the answer in the form shown.]

The above question might have been worded and worked in a different way, viz.

Arrange in descending powers of x

$$3x^2-2x+4+ax^2+4x-c-2x^2+bx+2.$$

$$\begin{aligned}\text{Expression} &= 3x^2-2x+4+ax^2+4x-c-2x^2+bx+2\\ &= 3x^2+ax^2-2x^2-2x+4x+bx+4-c+2\\ &= (3+a-2)x^2+(-2+4+b)x+(4-c+2)\\ &= (1+a)x^2+(2+b)x+(b-c).\end{aligned}$$

In working examples of this kind two points should be noticed:

(1) The first line of work should simply repeat the expression as in the question. **Take care to copy the expression exactly as it is in the question.** Many marks are lost in examinations through copying the question wrongly.

(2) The signs = should come one underneath the other, not in a slanting line.

36. Sometimes a straight line, called a **vinculum**, drawn above the terms, is used instead of a bracket. Thus $-\overline{7-ax}$ means the same as $-(7-a)x$.

The straight line of a fraction serves the same purpose as a bracket, e. g. $\frac{x-7}{3}$ is the same as $(x-7)\div 3$ or $\frac{1}{3}(x-7)$. It is very necessary to remember this when the minus sign is before the fraction.

Example. *Simplify* $8-\frac{x-3}{4}\times 12$.

$$\begin{aligned}\text{Expression} &= 8-\frac{x-3}{4}\times 12\\ &= 8-(x-3)\times 3 \quad \{\text{since 4 cancels}\}\\ &= 8-3x+9\\ &= 17-3x.\end{aligned}$$

Examples V a.

Simplify the following:

1. $3(5-4)-2(5-3)$.
2. $2\overline{7-x}+3\overline{2+5x}$.
3. $x(x-5)-x(x+4)$.
4. $12\times\frac{x-7}{3}-8\times\frac{x-3}{4}$.
5. $16(a+b-2c)+7(3a-b+2c)+8(2a-b-c)$.
6. $12(2x-y)-x(2-5)-y(7+3)$.
7. $\overline{a-b}\,x+\overline{b-c}\,x+\overline{c-a}\,x$.
8. $-p(q+r)-q(r+p)-r(p+q)$.
9. $(a-2b)c-(b-3a)c+(a+b)c$.
10. $(a-2b)c-(b-3c)a+(c+a)b$.

By graphic methods find the value of:

11. $(x+2)(x+3)$.
12. $(x-2)(x+3)$.
13. $(x+3)^2$.
14. $2x(a+b+c)$.
15. $(x+5)(x-5)$.

Arrange the following expressions in descending powers of x:

16. $3-2x+5-4x^3+2x^3-1$.
17. $2x^3-3x+4-x^2+7x^3+2x-5+x$.
18. $ax^3-3x+b-bx+cx^3+2x-5+ax$.
19. $p-qx+5x-px^3+qx^3-r$.
20. $(a+b)x^3+(b+c)x+(3-b)+(2-a)x+(2-b)x^2+a$.
21. Show graphically that $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$.
22. Simplify

$$6\times\frac{x-y}{2}+10\times\frac{2x-3y}{5}-12\times\frac{3x-5y}{6}.$$

Verify by putting $x=3$, $y=4$.

37. *Find the cost of* x *yards* y *feet of cloth at* x *shillings* y *pence per foot.*

Total number of feet $=3x+y$.

Each foot costs $12x+y$ pence.

$\therefore$ Total cost $=(3x+y)(12x+y)$ pence.

When two expressions in brackets are to be multiplied together the sign of multiplication is usually omitted, just as in the case of two single letters.

Now we could work the sum in a slightly different way.

One foot costs $12x+y$ pence.
$\therefore$ one yard costs $3(12x+y)$ pence.
And x yards cost $3x(12x+y)$ pence.
Also y feet cost $y(12x+y)$ pence.
$\therefore$ x yards y feet cost $3x(12x+y)+y(12x+y)$ pence.

Comparing this with the previous result, we see that

$$\begin{aligned}(3x+y)(12x+y) &= 3x(12x+y)+y(12x+y)\\ &= 36x^2+3xy+12xy+y^2\\ &= 36x^2+15xy+y^2.\end{aligned}$$

38. We have already shown that $c(a+b)=ca+cb$, where c may be any quantity we like.

Suppose that $c=x+y$, and substitute this value for c in the equality

$$c(a+b)=ca+cb.$$

We have $$\begin{aligned}(x+y)(a+b) &= (x+y)a+(x+y)b\\ &= ax+ay+bx+by.\end{aligned}$$

Putting this into words, we have the rule:—**When two expressions in brackets are to be multiplied together, the product is obtained by multiplying every term in one bracket by every term in the other bracket, and taking the algebraic sum of the products.**

The above process can sometimes be reversed, that is, if we have an expression without any brackets, we may be able to put in or insert brackets.

For $$\begin{aligned}ax+ay+bx+by &= a(x+y)+b(x+y)\\ &= (x+y)(a+b).\end{aligned}$$

Examples. $$\begin{aligned}3x+4y+3a+4b &= 3x+3a+4y+4b\\ &= 3(x+a)+4(y+b).\end{aligned}$$

$$\begin{aligned}3x+6y+x^2+2xy &= 3(x+2y)+x(x+2y)\\ &= (x+2y)(3+x)\\ &= (x+2y)(x+3).\end{aligned}$$

Examples V b.

Remove brackets in the following examples:

1. $(x+2)x$.
2. $3(x+2)$.
3. $(x+2)(x+3)$.
4. $(x-2)x$.
5. $3(x-2)$.
6. $(x-2(x-3)$.

7. $(x+5)(x+6)$. 8. $(x-3)(x+4)$. 9. $(x-a)(x-2a)$.
10. $(a-4)(a+2)$. 11. $(a+3)(b+2)$. 12. $(p-x)(2p+3x)$.

By graphical methods remove the brackets from

13. $(x+1)(x+2)$. 14. $(x-3)(x+2)$. 15. $(x-5)^2$.

Insert brackets in the following expressions:

16. x^2+3x. 17. $2x+6$. 18. $x^2+7x+12$.
19. x^2-3x. 20. $2x-6$. 21. $x^2-7x+12$.
22. x^2-ax. 23. $bx-ab$. 24. $x^2-ax-bx+ab$.
25. $ax-ab+ac$. 26. $a^2-ab-ac+bc$.

What expressions multiplied together give the following products?

27. $5x-10$. 28. $xy-2y-3x+6$. 29. $x^2+8x+16$.

30. A farmer leaves to each of his x sons 5 fields, and to each of his y daughters 3 fields. Each field contains $2x$ acres y roods. What was the area of the whole farm?

31. There are x houses on one side of a street, and y houses on the other. How many houses are there in the street? The rent of each house is £x ys., what is the total rental value of the street in shillings?

32. What is the area of a rectangle $(x+4)$ feet long, $(x-2)$ feet wide?

33. Each square of a chess-board is 1 square inch in area; the border is x inches wide. What is the area of the whole board?

34. A man sells x oxen and y sheep for $(2ax+2ay)$ pounds; what was the average price received for each animal?

35. A gives B £a, B gives C £b, C gives A £c, and then each has £x. How much had each at first?

39. A bracket may in several respects be compared to a parcel. Just as a person would rather do up half a dozen objects into one parcel than carry them separately, so in Algebra quantities are placed in a bracket when it is more convenient to deal with them as one quantity than to deal with them separately.

Again, the person would not undo the parcel unless he wanted something inside the parcel; similarly, in Algebra, a bracket should not be removed unless some term inside the bracket is particularly wanted. Usually terms containing x are wanted.

Example I. *Arrange in ascending powers of* x *the following expression,* $(ax+b)c+(a+b)x+bx(x^2+cx)$.

Expression $= (ax+b)c+(a+b)x+bx(x^2+cx)$.
$= acx+bc+(a+b)x+bcx^2+bx^3$.
$= bc+x(ac+a+b)+bcx^2+bx^3$.

Here the first and third brackets are removed because x is inside them, but the second is kept because x is not inside it.

Example II. $(x+a)(x+b) = (x+a)x+(x+a)b$
$= x^2+ax+bx+ab$
$= x^2+(a+b)x+ab$.

Here the brackets are first removed because x is inside them, and then a bracket is inserted in order to get one term $(a+b)x$ instead of two terms $ax+bx$.

40. A parcel may contain another parcel, the second parcel may contain a third parcel, and so on. In the same way a bracket may contain other brackets. When this is the case brackets of different shapes are used to avoid confusion.

Example. *A certain number* x *is subtracted from* 10, *the remainder is subtracted from* 7. *This second remainder is multiplied by* 5 *and the product added to the original number. This last result is squared; find an expression for the final result.*

First remainder $(10-x)$.
Second remainder $\{7-(10-x)\}$.
Multiply by 5 $5\{7-(10-x)\}$.
Add to x $[x+5\{7-(10-x)\}]$.
Square $[x+5\{7-(10-x)\}]^2$.

It is not often that more than the three kinds of brackets here shown are wanted. If they should be wanted, the same shapes are used again, but made larger. The necessity for more brackets is often avoided by using a vinculum.

When brackets are to be removed, it is usually more convenient to remove the innermost bracket first, and to work outwards, but it is quite possible to remove the outermost first and to work inwards.

The brackets in the above question could be removed in either of the following ways:

(i) Working from within

$$\begin{aligned}[x+5\{7-(10-x)\}] &= [x+5\{7-10+x\}] \\ &\qquad \text{removing inside bracket,} \\ &= [x+5\{-3+x\}] \qquad \text{collecting,} \\ &= [x-15+5x] \\ &= 6x-15;\end{aligned}$$

$$\begin{aligned}\therefore [x+5\{7-(10-x)\}]^2 &= (6x-15)^2 \\ &= (6x-15)\,(6x-15) \\ &= (6x-15)\,6x-15\,(6x-15) \\ &= 36x^2-90x-90x+225 \\ &= 36x^2-180x+225.\end{aligned}$$

(ii) Working from without

$$\begin{aligned}[x+5\{7-(10-x)\}] &= x+5\{7-(10-x)\} \\ &= x+35-5\,(10-x) \\ &= x+35-50+5x \\ &= 6x-15;\end{aligned}$$

and the work would be finished as in method (i).

Examples V c.

Remove the brackets and simplify:

1. $12a-[6a-2\{3a-4(b-a)\}-(9a+8b)]$.
2. $3a-[2b-\{4c-12a-(4b-8c)\}-(6b-12c)]$.
3. $3a-[5b-\{6a-(4a-7b)\}]$.
4. $x-\{2(y-z)+3z\}$.
5. $4(x+y)+2[x+3\{y-2z-(2x+y)\}]$.
6. $-[2x-\{y-z-(3x-y)\}]$.
7. $-[2y-\{z+y-(3z+2x)\}]$.
8. $6a-2[b-4(3c-2x)+3\{a-(4c+x)\}]$.
9. $3p-[q-3\{p+2(q-p)\}-(3q+p)]$.
10. $[2a+b-\{4a-(5a-\overline{a-2b-c})\}]+[a-\{a-(a-\overline{a-b})\}]$.
11. If $a=-10$, $b=8$, $c=1$, find the value of

$$2a-3b+c-[-8a-\{2c-4b+(5b-a)\}] \\ -[4a+\{b-c-(2c+b-a)\}]$$

12. If $a=-3$, $b=2$, $c=0$, find the value of

$$2b-a[-a+\{b-2c(b-a)\}+b].$$

13. Simplify $6\left\{\frac{x-7}{3}-2\left(x+\frac{x+4}{12}\right)-x\right\}$.
14. Simplify $2a-\{3a-(4b+2a)\}+\{5a-(4b-a)\}$.

15. If $a = 3$, $b = 1$, find the values of
(i) $2a^3-(3ab-b^3)$; (ii) $2a^3-3(ab-b^3)$; (iii) $2\{a^3-3(ab-b^3)\}$.

16. Find the value of
$[a-2\{b-(c-d)\}]\,[3a-\{b+2(c+d)\}]$
$-[a-b+3(c-d)]\,[3a+b-2(c-2d)]$,
when $a=1$, $b=2$, $c=-1$, $d=-3$.

17. Simplify $2x-[y-2\{x+(y+x)+2y\}+2x]$.

18. Simplify $3\{(3a-b)^2+(a-3b)^2\}-10(3a-b)(a-3b)$ and verify the result when $a=2$, $b=1$.

19. Simplify $3(a+c)-5(a+b+c)-[b-\{c+2a-(b-c)\}]$.

20. Find the sum of $[-5x+7y-\{5x-(6x+7)-(4y+1)\}]$ and $-[-(4x+7y)+\{8-(-8x+9y)+(5x-8y)\}]$.

Examples V.

1. Find the value of $a+b-[a-\{c+2a+b+(2c+b)\}]$ when $a=3$, $b=4$, $c=2$.

2. Subtract $2x-3(y+2z)$ from $3(x+y)-8z$.

3. Add together $a-\{b+(c-2d)\}$, $2(3a+b)-4(c+d)$, and $5d+3\{c-(2a+b)\}$.

4. Subtract $q-[q-\{q-(q-\overline{q-p})\}]$
from $p-[p-\{p-(p-\overline{p-q})\}]$.

5. Add together $a-\{b+c-(a+b)\}+c$, $2(3a+2b)-4(b+2a)-c$, and $3(2b-a)-2(3b-a)+c$.

6. Find the remainder when $3a-5\{a-b-(c-2b)\}+b$ is taken from $c-2[c-\{a-2(c-b)\}]$.

7. Subtract $(a-b)(b+c)$ from $(b-c)(c+a)$.

8. Simplify $x(2y-3z)-2y(z-3x)+3z(x-2y)$.

9. Simplify $-7[-11x-4\{-17x+3(5x-1)\}-3(x+4)]$.

10. Arrange $a(b^3-c)+b(c^3-a)+c(a^3-b)$ in descending powers of c; and find the value when $a=2c$, $b=3c$.

11. Simplify $7x^3-(3-4ax)-3x(4x+1)(ax+5)$; and re-bracket the result according to descending powers of x.

12. What is the value of $a(x^3-ay^3)+xy(x-a^3y)$ when $x=ay$?

13. A dealer buys p horses for £a each; he sells q of them for £b each, and the rest for £c each. Find an expression for his gain on the transaction.

14. A man's salary is £x a day and he spends £y a day. How much does he save in z days?

15. A house contains p rooms on the ground floor and q rooms on the first floor. In each room there are p pictures on each

short wall and q on one of the long walls; how many pictures are there altogether?

16. In a certain field there are x cows and y sheep; the average value of each animal is x pounds y shillings. Find the total value of the animals.

17. A big box contained 50 apples and a large basket. In the basket there were 38 apples, a paper bag, and a cardboard box. The bag contained 6 apples; in the cardboard box there were 5 apples and a bag containing 8 apples. How many apples were there altogether?

18. If $y = 2x - 3$, express $3x + 5y$ in terms of x only.

19. If $y = 3x + 2$, what are the values of y when $x = 0, 1, 2, 3$ respectively? What value of x will make $y = 14$? Is there any value of x which makes y equal to x?

20. If $y = 3x + 2$ and $z = 3y - 4$, express z in terms of x. Find the value of z when $x = -3$ by two methods.

CHAPTER VI

EASY FUNCTIONS AND THEIR GRAPHICAL REPRESENTATION

41. If a quantity continually changes its value it is called a **variable**; a quantity which does not change its value is called a **constant**. x, y, z are usually used to denote variable quantities; a, b, c are used for constant quantities. If a man is running, the time he has been running and the distance he has run are continually changing—they are variables; but his speed may be the same throughout; in such a case speed would be a constant.

42. Very often two variables are so related to one another that if one is known the other can be found.

Example. *A man spent £x in buying* y *cricket balls at 5 shillings each.*

Here $20x$ = number of shillings spent
and $5y$ = number of shillings spent.
$\therefore\ 5y = 20x.$
Divide by 5. $y = 4x.$

Hence if x is known, y can be found. In such cases as this x is called the **independent variable** and y the **dependent variable.**

43. Example. *If* $y = 3x^2 - 2x + 5$, *make a table showing the values of* y *corresponding to the integral values of* x *between* -3 *and* $+3$.

If $x =$	-3	-2	-1	0	1	2	3	4
then $x^2 =$	9	4	1	0	1	4	9	16
$3x^2 =$	27	12	3	0	3	12	27	48
$-2x =$	6	4	2	0	-2	-4	-6	-8
and $y =$	38	21	10	5	6	13	26	45

Any expression whose value depends on the value of x is called a **function of** x. Thus $4x$, $3x^2 - 2x + 5$, $3 + \frac{2}{x}$ are all functions of x.

Examples VI a.

Make tables showing the values of the following functions as x changes from -2 to $+5$.

1. $y = 3x + 2.$
2. $y = 2x - 4.$
3. $y = 3x^2 - 2x + 1.$
4. $y = 2x^2 + 4x - 5.$
5. $y = x^2 - 4x + 3.$
6. $y = x^3 - 2x.$
7. $y = 3 + 2x.$
8. $y = 3 + 2x^2.$
9. $y = 4 - x.$
10. $y = x - x^2 + 2.$

GRAPHICAL REPRESENTATION OF FUNCTIONS

44. It has been shown in § 19 that there are two ways of representing a quantity graphically; lines may be drawn horizontally, i. e. to the right or left, or vertically, i. e. up or down. By using both methods we can represent two variables in the same figure.

Example. *To show in a figure that when* $x = 4$, $y = 7$.

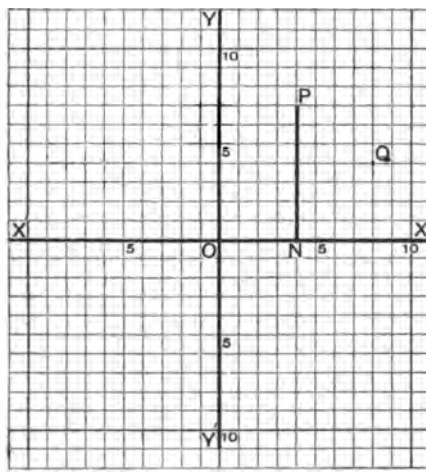

FIG. 15.

Using squared paper take any point O where two lines cross; see Fig. 15.

Measure $ON = 4$ to the right, then ON represents $+4$.

From N measure NP upwards, then NP represents $+7$.

It is always understood that horizontal lines represent x and vertical lines represent y; hence any one looking at Fig. 15 can see at once that when $x = 4$, $y = 7$.

The line NP is not always actually drawn; it is sufficient to count the squares and to make a dot or small cross when the required numbers have been counted. Thus in Fig. 15 the point Q shows that when $x = 8$, $y = 4$.

When values of x and y are represented in this manner they are said to be **plotted.**

45. The point O is called the 'origin', because it is the point from which the measurements start. When the point O has been chosen, the horizontal and vertical lines are drawn to show in what directions the measurements are to be made. These lines are called the axes; the horizontal line is the x-axis and is named $X'OX$, the vertical line is the y-axis and is named YOY'.

The next step is to choose the scale. When fractional values do not occur it is usually convenient to take the side of 1 small square to represent 1 unit.

Having chosen the unit, mark off every fifth unit along the axes. Except when the side of a square represents the unit, the scale should be written in one of the corners of the figure, as has been done in previous sections.

Recollect that negative values of x are measured to the left and negative values of y are measured downwards.

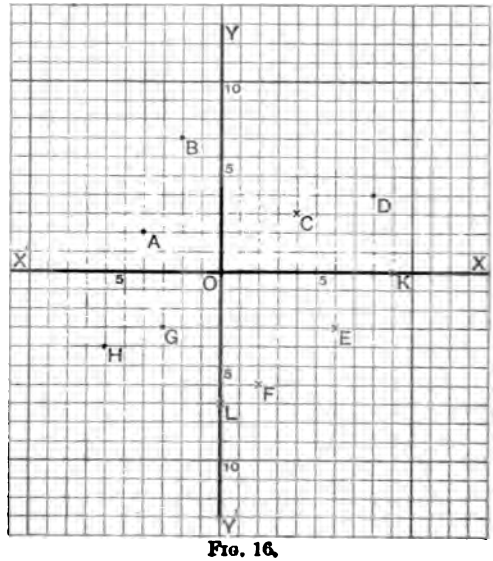

FIG. 16.

Examples VI b.

1. Show graphically in one figure that:

when $x =$	3	4	5	-1	-2	-7	-9	6
then $y =$	-4	-7	2	$+3$	-6	$+3$	-9	5.

2. What relations between x and y are represented by the points $A, B, C, D, E, F, G, H, K, L$ in Fig. 16?

3. Represent in a diagram the values of x and y given in the table of § 43.

46. The graphical method described in the last section is often used to show how quantities change from time to time. A very common example is the diagram that appears in newspapers to show the height of the barometer at different times during the day. In making these diagrams, the lines that are to represent time are generally measured along the x-axis, the other quantity being represented by lines perpendicular to the x-axis.

Example. *During a certain week the temperatures at noon are given by the following table:*

	Sun.	*Mon.*	*Tues.*	*Wed.*	*Thurs.*	*Fri.*	*Sat.*
Temp.	29°	35°	42°	31°	27°	43°	50°

Represent this graphically.

1. As there are no negative quantities, measurements may start from a point near the left-hand bottom corner instead of from a point in the middle of the figure, see Fig. 17.

2. As no temperature is less than 27°, it will save space if we begin counting the temperatures at 25°, i. e. 27° will be 2 units up, not 27.

3. In order not to cramp the figure, take twice the side of a square to represent a day.

4. When all the points have been put in, join up the consecutive points by straight lines.

The broken line formed by joining the points is called a **graph.**

The graph is shown in Fig. 17; it tells us all that the table of the question does, and also shows at a glance which day was the coldest at noon, which day had the lowest

average temperature, which day had the greatest change of temperature, &c. But this graph does not enable us to find the temperature at any other times than noon on the respective days.

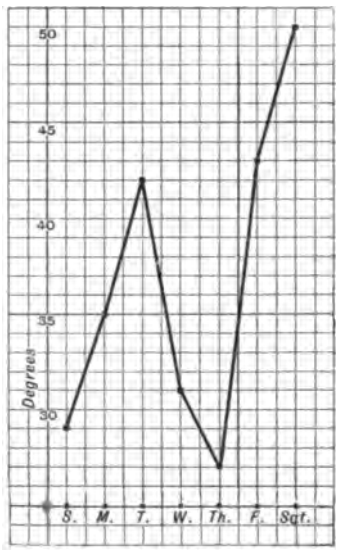

FIG. 17.
Scale: side of square = 1°.

If, however, the temperatures had been taken every half-hour the points on the diagram would have been much nearer together, and if the same scale had been used the points might be joined by a curved line instead of by a series of straight lines. In such a case the graph would give a very good idea of what the temperature was between the readings, as well as what it was at the actual times of the readings.

Examples VI c.

Draw graphs from the following tables:

1. A boy's position in form is given by the following table:

Date	Jan. 30.	Feb. 6.	Feb. 13.	Feb. 20.	Feb. 27.	Mar. 6.
Place	5	12	17	6		15

Draw a graph. Can you draw any conclusion as to his place on February 27?

2. In consecutive cricket matches a boy made the following scores: 17, 4, 0, 6, 3, 12, 1, 2, ?, 7, 3.

Draw a graph. Can you draw any conclusion as to the missing score?

3. A thermometer was placed in some water which was then heated, and the temperature was read every minute. Draw a graph from the following table:

Temperature in degrees	12	16	19·5	22·5	25	26·5	27
Time in minutes	0	1	2	3	4	5	6

Would this graph give any information as to the temperature after $4\frac{1}{2}$ minutes?

4. If $y = 3x - 2$, calculate the values of y when x has the integral values from -5 to $+5$; and draw the graph.

From the graph try to find the values of y when $x = -\cdot5$ and $x = +\cdot25$. Check the answers by actual calculation.

5. A man sells kettles. He has only made them of three sizes as yet, and he has fixed on the following as fair list prices:

Capacity in pints	12	6	2
Price in pence	68	50	22

He knows that other sizes will be wanted, and he wishes to publish at once a price list for many sizes. By drawing a graph obtain approximate prices for 4- and 8-pint kettles.

6. The keeper of a restaurant finds that when he has G guests a day his total daily profit is P pounds. The following numbers are averages obtained by comparing many days' accounts. Draw a graph, and so obtain approximately the number of guests for which there would be neither gain nor loss.

G	210	270	320	360
P	−0·9	+1·8	+4·8	+6·4

7. A train leaves Manchester at 6 a.m.; the following table shows the distance it has travelled at certain times. Draw a graph, and find (i) the approximate time it takes to travel 9 miles, (ii) how far it has travelled in 40 mins.

Distance in miles	1	4	5	10	12	16	18
Time	6.4	6.11	6.16	6.28	6.36	6.47	6.52

8. The following table shows the time for lighting cycle lamps on certain days in a certain month:

Date	1	2	5	8	11	13	17	20	24	28
Time	5.46	5.48	5·53	5.57	6.3	6.7	6.15	6.20	6.28	6.35

Draw a graph, and find the lighting-up time on the 4th, 15th, and 22nd.

9. The height of the barometer at noon, on certain days in June, is given in the following table:

Height	30·13	30·10	30·10	30·23	29·96	29·83	29·43
Date	1	2	3	4	5	7	8

Draw a graph, and if possible find the height on the 6th.

10. The amateur records for certain distances are given in the following table. Represent them in a diagram.

Distance in yds.	100	120	150	200	250	300	440
Time in secs.	$9\frac{3}{5}$	$11\frac{1}{5}$	$14\frac{4}{5}$	$19\frac{1}{5}$	$24\frac{1}{5}$	$30\frac{3}{5}$	$47\frac{1}{5}$

11. Represent in a diagram the winner's time in the University Boat Race as given in the table below.

Year	1890	1891	1892	1893	1894	1895	1896
	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.
Time	22 3	21 48	19 21	18 47	21 39	20 50	20 2

12. The following table shows the number N of articles that may be bought for £1 when the price is x pence :

x	5	6	8	10	12	20	24	30	40	48
N	48	40	30	24	20	12	10	8	6	5

Draw a graph. How does this graph differ from the previous graphs ?

13. In a price list the following prices are given for tree pruners :

Length in feet	2	4	6	8	10
Price	3*s.* 6*d.*	4*s.* 3*d.*	5*s.*		6*s.* 6*d.*

Represent these prices by a graph, and try to find the price of a pruner 8 feet long.

14. The price of dish covers is given in the following table. By means of a graph find the approximate price of a 12-inch cover.

Length in inches	10	12	14	16	18
Price	9*s.* 3*d.*		15*s.* 9*d.*	20*s.*	26*s.* 3*d.*

47. Graphs of algebraical functions.

Example. *Draw the graph of* y *when* $y = x^2 - x$.

Step 1. Tabulate the values of y as in § 43.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	
x^2	25	16	9	4	1	0	1	4	9	16	25	36	
y	30	20	12	6	2	0	0	2	6	12	20	30	

Step 2. Choose the scale. This depends to a great extent upon the size of the piece of squared paper that is to be used. If we wish to graph all the above values, the paper must be big

enough to show $y = 30$; as there are no negative values of y, about 35 squares in length will allow us to take the side of 1 square to represent 1 unit. In Figs. 18 and 19, the points that

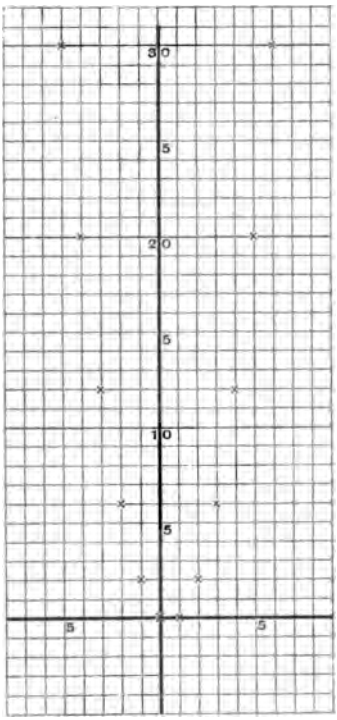

FIG. 18.
Scale: side of 1 square = 1 unit.

represent the values for x and y obtained in Step 1, are shown by crosses. Sometimes it is necessary to take different scales for x and y. In this example, if the squared paper were a small piece, it might be advisable to take the side of a square to represent two units for y, but only half a unit for x. This is the scale used in Fig. 19.

Step 3. Should the graph be finished by joining consecutive points by straight lines, or ought a curve to be drawn passing

through all the points? A little consideration shows that a slight alteration in x will produce only a slight alteration in the value of y. In all such cases a curve should be drawn, provided sufficient points have been obtained to make the shape of the curve quite clear. In Fig. 19 the points are closer than in Fig. 18, and the form of the curve is more obvious. In both curves there is no indication

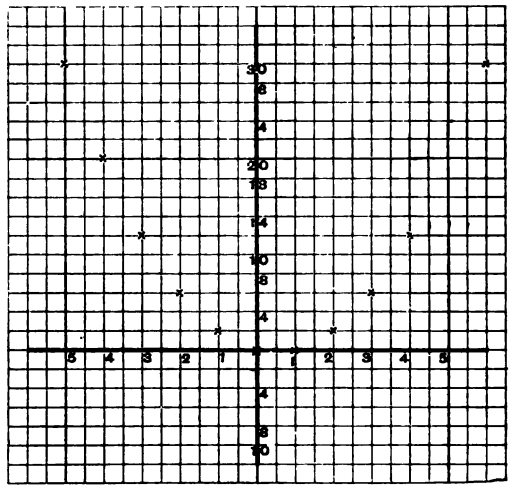

FIG. 19. Scale: 2 sides = 1 for x; 1 side = 2 for y.

of the shape of the curve between $x = 0$ and $x = 1$. We must therefore find the values of y corresponding to values of x between 0 and 1. With a small scale it is no use working the values of y correct to more than two decimal places.

x	·2	·4	·5	·6	·8
x^3	·04	·16	·25	·36	·64
y	−·16	−·24	−·25	−·24	−·16

It is now seen that the least value of y is $-\frac{1}{4}$, and that this corresponds to $x = \frac{1}{2}$. This point must be found as accurately as possible, the sides of the square being divided by eye measurement.

Step 4. The graph should be finished by drawing a curve through all the points obtained. It should be a graceful curve without any awkward bends.

48. It often happens that only a portion of a graph is actually wanted; when this is the case, a larger scale can be used. Suppose the graph of $y = x^2 - x$ were required only from $x = 0$ to $x = 1$. The values of x and y have been obtained in the last section. We can now take the side of a square to represent only one-tenth of a unit. The resulting graph is shown in Fig. 20.

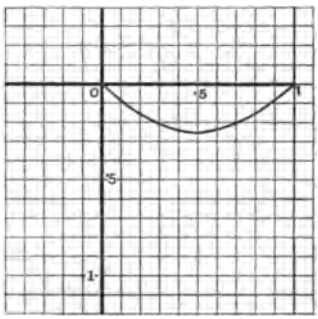

FIG. 20. Scale: 1 side = $\frac{1}{10}$ unit.

Examples VI d.

Draw the graphs of the following functions, and in each case find the value of y when $x = 3{\cdot}5$ and verify by calculation:

1. $y = x$.
2. $y = x^2$.
3. $y = x^3 + x$.
4. $y = x + 2$.
5. $y = 2x + 3$.
6. $y = 3x^2 - 4$.
7. $y = 3 - 4x$.
8. $y = \frac{x^3}{4}$.
9. $y = 2x^2 - 5x + 6$.
10. $y = 7x - 9$.

In the following graphs find the value of x when $y = 2{\cdot}7$ and verify by calculation:

11. $x = 2y + 3$.
12. $x = 3y - 5$.
13. $x = y^2$.
14. $x = 3y^2$.
15. $x = 3y^2 + y$.
16. $x = y^2 - 2y + 4$.
17. $x = 3 - y^2$.
18. $x = 4 - 2y$.
19. $3x - 4y = 1$.
20. $4x + 5y = 20$.

49. Some of the graphs in Examples VI d are curves,* some are straight lines. It should be noticed that when the function is of the first degree (i. e. containing only the first power of the independent variable and no fractions with the variable in the denominator) the graph is a straight line. This is found to be the case always, and so we learn by experience that

(1) The graph of a function of the first degree is a straight line. For this reason functions of the first degree are often called linear functions.

(2) The graph of a function that is not of the first degree is not a straight line.

As a straight line can be drawn when two points on it are known, we need find only two points when drawing a graph of a function of the first degree. To ensure accuracy the following directions should be noted:

1. Plot three or four points before drawing the straight line. For a slip may have been made in calculating the first two; when the third is plotted it will then be seen that it is not in the same straight line as the others; on plotting the fourth it will probably be found to be in the same straight line as two of the others and the line can be drawn. If no three of the points are on a straight line two or more mistakes have been made, and the previous work should be done over again.

2. The two points actually joined should not be too close together; for if the points are not joined accurately the resulting error is less when the points are far apart.

3. The values of x and y which are plotted should be whole numbers, or should be such that the point representing them is at the intersection of two of the lines on the squared paper. For if not the positions of the points cannot be found exactly and the straight line is certain to be somewhat inaccurate.

* The word curve is often used to include all lines whether straight or bent; it will be used in this general sense throughout the book. The context usually makes it clear whether the word includes straight lines or not.

Example. *Draw the graph of* $y=\frac{3}{5}x-\frac{2}{5}$.

This is of the first degree.

Therefore the graph is a straight line and we need find only two points.

$$y=\frac{3x-2}{5}$$

x	-6	4	9
$3x-2$	-20	10	25
y	-4	2	5

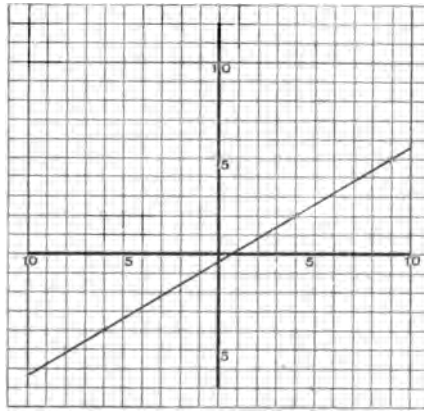

FIG. 21. Scale : 1 side = 1 unit.

Graphs from the Point of View of Geometry.

50. Hitherto we have regarded graphs merely as showing how the value of a function y varies when the value of x changes ; and we have seen that a point can be found which represents any pair of corresponding values of x and y. If we wish to find the point we have had to find the values of x and y.

The subject of graphs is often introduced in the reverse way as being an aid to Geometry.

If the position of a point is known, the point can always be found by taking two measurements.

Geography supplies useful examples. If an explorer finds a mountain that has been previously unknown, he calculates the latitude and longitude of the mountain, with the result that the mountain can afterwards always be found.

In Geometry the measurements taken are (1) the perpendicular distance of the point from a fixed straight line $X'OX$, (2) the distance of the foot N of the perpendicular from a fixed point O on the axis. This point O is shown by drawing a line $Y'OY$ perpendicular to $X'OX$. The length ON, NP are denoted by x and y respectively, the usual rules for signs being observed.

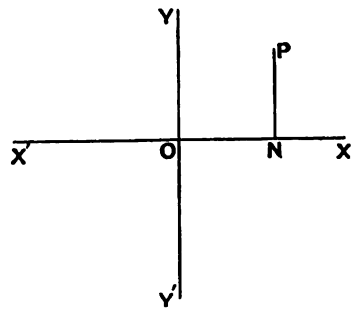

FIG. 22.

x and y are called the **co-ordinates** of the point P. x, the piece of the axis of x 'cut off', is called the **abscissa**. y, which shows in what 'row' or 'rank' the point lies, is called the **ordinate**.

A point whose co-ordinates are $x = 5$, $y = 6$ is referred to as the point (5, 6).

51. If a point moves so as to satisfy some condition it describes a curve which is called in Geometry the 'locus of the point'. As the point moves its co-ordinates change, but, if the point satisfies some condition, its co-ordinates will also satisfy some condition. Conversely, if the co-ordinates are always connected by some relation, the point will always

lie on some curve. In Examples VI d various relations between x and y are given, and the point must lie on the corresponding graph.

Examples VI e.

1. Plot in one figure the following points:
$(4, 5)$, $(-3, 6)$, $(6, -4)$, $(0, 7)$, $(-5, 0)$, $(-6, -5)$.

2. A point moves so that its ordinate is always 2; draw the graph traced out.

3. Plot six different points all having the abscissa equal to 3. Hence find the graph of $x = 3$.

Draw the curve traced out by a point which moves so that its co-ordinates are connected by the relation:

4. $y = x$.
5. $y = 3x - 2$.
6. $y = \frac{4x-1}{3}$.
7. $y = 4x - 1$.
8. $y = 4x + 3$.
9. $y = x^2 - 9$.
10. $y = x^3 - x^2$.

CHAPTER VII

SYMBOLIC REPRESENTATION AND SIMPLE PROBLEMS

52. We have already had many examples of symbolic representation, and the student is by now quite used to the idea that letters may be used as symbols for numbers, and that the operations of addition, subtraction, &c., are represented by the special symbols or signs $+$, $-$, &c.

It must be recollected that letters are used as symbols for numbers, and, if the work deals with concrete quantities, the units must always be mentioned; thus we should not say 'a man's age is x' but 'a man's age is x years'.

Letters may be used instead of numbers for three different reasons:

(i) To state facts that are true about all numbers, e.g. $a + b = b + a$ states the fact that, whatever two numbers we take

the first added to the second is equal to the second added to the first.

(ii) To stand for some particular numbers whose values are unknown. The letters x, y, z are often used to stand for numbers which are unknown, but whose value is to be found by means of information given in the question. Questions of this kind, which require that a certain unknown number should be found, are called **problems.**

(iii) To state concisely rules for calculating certain concrete quantities. Thus it was shown in § 11 that the area of a rectangle is obtained by multiplying the number of units of length in one side by the number of units of length in the adjacent side. This is stated more concisely thus: $A = lb$ where A stands for the number of units of area, l for the number of units of length in the length, and b for the number of units of length in the breadth.

A concise rule of this kind is called a **formula.** It is usual in formulae to use letters which suggest what they stand for, e.g. l for number of length units, t for number of time units, &c.

53. In working the following examples the student should read each question carefully, and make sure that he understands it before attempting to answer it. If he cannot see how to start, let him mentally replace the letters by small numbers, so making a simple arithmetic sum. Then he should do with the letters exactly the same as he would with the numbers.

Even though the whole question can be worked in the head, yet sufficient work must be put on paper to show clearly the line of argument.

Example. *A man can buy* x *articles for a certain sum, the price being* a *pence each. How many would he get for the same sum if the price were* $(a+b)$ *pence each?*

[Mental work. A man buys 10 articles at 3*d.* each; how many would he get for the same sum if the price were $(3+2)$ pence each?

Total sum spent $= 10 \times 3 = 30$ pence.

If one cost 5 pence, the number would be $30 \div 5 = 6$.]

Written work.

Total sum spent $= x \times a = ax$ pence.

If one article cost $(a+b)$ pence, he would get for the same sum $ax \div (a+b) = \frac{ax}{a+b}$ articles. *Answer.*

Examples VII a.

1. What is meant by an even number? Prove that the square of any even number is exactly divisible by 4.

2. Find a formula which will show the connexion between the length of the side of a square measured in yards and the area of the square in acres.

3. If a man earns x shillings every week-day, what is his yearly income in pounds?

4. A boy was born in the year A.D. x; how old will he be in the year A.D. y?

5. Prove that when two odd numbers are multiplied together the product is odd.

6. If a train travels u miles in every hour, and if it travels s miles in t hours, find the formula connecting u, s, and t.

7. A man has £x in his pocket, his wife has x shillings, and each of their three children has x pence; how much money have they between them? Express the answer in shillings.

8. During the cricket season a boy scored x runs in y innings and was not out z times. What was his average?

9. How many days will it take a steamer travelling at the rate of v miles per hour to travel s miles?

10. A room is l feet long, b feet broad, and h feet high; find the area of the floor and the area of the walls.

11. How many numbers between 10 and 100 are exactly divisible by 7? Write down the first, second, third, and the rth of those numbers.

12. What is the general algebraic form of a number which can be exactly divided by 5? What do you know about the number $5x+3$?

13. A score of eggs cost x pence. What is the price of y eggs?

14. A cycles at the rate of x miles per hour, B walks at the rate of y miles per hour. How many miles does A gain each hour? If B starts z hours before A, how long will A take to catch B?

15. A workman's wages are a pence per hour for b hours in the week of regular time and c pence per hour for overtime. One week he earns d shillings. How many hours overtime did he work?

16. Prove that the sum of any two numbers multiplied by their difference equals the difference of their squares.

17. What is the value of x sovereigns $2x$ shillings and $(20-3x)$ sixpences? Give the value in sixpences.

18. A garrison have food enough to last 1000 men x days; how long will the food last if the number of men is increased to 3000?

19. If I buy a apples at p pence a dozen and b apples at q pence a dozen, and sell them all at one penny each, how much profit do I make?

20. A room is l feet long, w feet wide, h feet high. What will it cost to whitewash the walls and ceiling at p pence per sq. ft.?

21. A sum of P pounds is invested at simple interest at r % per annum. Find a formula for the interest after t years.

{r % per annum means that the interest on £100 for 1 year is £r.}

22. If x lb. of tea cost y shillings, how many shillings will y lb. of the same tea cost?

23. What should be the price of 7 eggs when a dozen cost x pence?

24. If x ounces of silver are worth y shillings, how many pennyweights of silver are worth z pence?

25. I bought $10x$ yards of cloth at y shillings per yard and sold them at $y-4$ shillings per yard; how many pounds did I lose?

26. If a clock gains x seconds in one day, how many minutes will it gain in y weeks?

27. Prove that the sum of three consecutive numbers is always divisible by 3.

28. If each cubic foot of a substance weighs w pounds, what is the weight of v cubic feet?

29. A body whose volume is v cubic centimetres weighs w grammes, and the average weight of a cubic centimetre is d grammes. What is the formula that connects v, w, and d?

30. If 100 degrees on the Centigrade thermometer are equivalent to 80 degrees on the Réaumur thermometer, how many Centigrade degrees are equal to x Réaumur degrees?

31. Find a formula for all numbers which give remainder 5 when divided by 7. Write down all those numbers between 150 and 200.

54. In problems the question enables us to state that two different expressions are equal to one another; and then by using ordinary common sense the unknown quantity can very often be easily found.

Example I. *A man is three times as old as his brother. In ten years' time he will be twice as old. How old is the brother now?*

Let x years be the brother's age now;
then $3x$ years is the man's age now.
Again $x+10$ years will be the brother's age in 10 years' time;
and $3x+10$ years will be the man's age in 10 years' time.
But the question says that the man's age in 10 years' time will be twice the brother's age.

$\therefore$ $3x+10 = 2(x+10)$.
Remove bracket $3x+10 = 2x+20$.
Take away 10 from each side $3x = 2x+20-10$.
Take away $2x$ from each side $3x-2x = 20-10$,
$x = 10$.

i. e. the brother's present age is 10 years.

Example II. *A sum of two guineas is divided between three men so that the first receives eight shillings more than the second and two shillings less than the third. How much did each receive?*

[Here there are three unknown quantities, but if we find what the first man receives we can easily find what the other men receive.]

Let x shillings be the amount the first man receives;
then $(x-8)$ shillings is the amount the second man receives,
and $(x+2)$ shillings is the amount the third man receives;
then $x+(x-8)+(x+2)$ shillings is the sum distributed.
But the question says two guineas, i. e. 42 shillings is the sum distributed.

Hence $x+(x-8)+(x+2) = 42$.
Remove brackets $x+x-8+x+2 = 42$.
Collect the x terms and the numbers
$3x-6 = 42$.

Add six to each side

$$3x = 42 + 6, \quad \text{since } -6 + 6 = 0;$$

i.e. $3x = 48,$

$$x = 16.$$

First man receives 16*s*., second man 8*s*., third man 18*s*.

In solving problems

(i) **Make sure what the question tells us.**

(ii) **Make sure what it is required to find.**

(iii) **Recollect that x stands for a number only, and that the unit must be distinctly stated, e. g. let x feet be the length, let x be the number of men, let x shillings be the money.**

Examples VII.

1. Find a number which is 18 less than double the number.

2. Find a number such that half the number is 5 more than one-third of the number.

3. The sum of three consecutive numbers is 57; find them.

4. A boy has 6*d*. less than his brother, and 9*d*. more than his sister; together they have 2*s*. 9*d*. How much has the sister got?

5. Three stations, *A*, *B*, *C*, are on the same line of railway. The distance from *A* to *B* is twice the distance from *B* to *C*, and the distance from *A* to *C* is 126 miles. How far is it from *A* to *B*?

6. The number of square yards in the area of a certain room is double the number of feet in the length of one of the sides. What is the length of the other side?

7. Of two numbers, one is three times as great as the other. If I take the less number from 16 and add 3 the result exceeds by 7 what I obtain when I subtract the greater number from 30. Find the two numbers.

8. A bill of £12 is paid in sovereigns and half-sovereigns, and 17 coins are used. How many sovereigns are there?

9. A father's age is 45 years and his son's 9. In how many years will the father be three times as old as the son?

10. A sum of £100 was contributed by *A*, *B*, and *C*. *A* contributed £5 more than *B*, and *C* £3 less than *A*. How much did each contribute?

11. Fourteen years ago a man was twice as old as his son;

at the present time their united ages amount to 100 years. Find the present age of each.

12. The length of a rectangular field is twice its breadth; the length of fence used to enclose the field is 372 yards. Find the area of the field.

13. A post stands with half its length in earth, one-third in water, and 4 feet above the water. How long is the post?

14. A man has in his right-hand pocket three times as many shillings as in his left. If he takes 17 shillings from his right pocket and puts them into the left, he will then have 6 times as many in the left-hand pocket as in the right; how many shillings were there in each pocket at first?

15. Find three consecutive numbers such that three times the greatest shall be equal to twice the sum of the other two.

16. Twenty years hence I shall be seven times as old as I was 28 years ago; how old am I now?

17. Divide £1000 into three parts, so that the first may exceed the second by £350 and the third may be one-half the sum of the other two.

18. A father's age is three times that of his son. Eleven years ago the father was five times as old as his son. What are their ages?

19. The cost of 7 ducks and 11 chickens is 55*s.* and the cost of 13 ducks and 5 chickens is 52*s.* What is the cost of a chicken?

20. A purse contains £9. 10*s.* in sovereigns and half-crowns, and the total number of coins is 20. Find how many sovereigns there are in the purse.

CHAPTER VIII

EQUATIONS

55. In previous examples we have frequently had to find the particular value of x that will make two expressions equal. Questions of this kind occur so often that we must now discuss the usual methods for answering them.

Definitions. An equation is a statement that two expressions are equal.

An equation may be true for all values of the letters, e. g. $(x+5)-6 = 2x-(x+1)$ is true for all values of x, since each side is only another form for $x-1$.

Such an equation is called an **identical equation** or **identity**. Very often the sign $\equiv$ is used instead of $=$ when it is known that the equation is an identity.

But usually an equation is true only for some particular value or values of the letters. As such equations are true on certain conditions, they are called **conditional equations**. Unless it is otherwise stated, the word 'equation' may be taken to refer to a conditional equation.

Consider $x-2 = 10$. This equation is certainly not true when $x = 4$, for then $x-2 = 4-2 = 2$; and it will be found that the equation is true only when $x = 12$.

The process of finding for what value of x the equation is true is called **solving** the equation.

A value of x which makes the equation true is called a **root** of the equation, and is said to **satisfy** the equation. The letter whose value is sought is often called the **unknown quantity**.

A **simple**, or **linear**, **equation** is one which contains only the first power of the unknown quantity.

56. Axioms.* 1. If equals be added to equals, the sums are equal.

2. If equals be subtracted from equals, the differences are equal.

3. If equals be multiplied by equals, the products are equal.

4. If equals be divided by equals, the quotients are equal.

5. Things which are equal to the same thing are equal to one another.

By continued application of these axioms any equation may be solved.

* Statements which are obviously true, which can be proved only by experience, and which are useful to prove other statements, are called **Axioms**.

Example. *Solve the equation* $3x-2 = 8-2x$.

Our object is to reduce this equation to the form $x =$ a number.

$$3x-2 = 8-2x.$$

Add 2 to each side $3x = 8+2-2x$, since $-2+2 = 0$.

Add $2x$ to each side $3x+2x = 8+2$, since $-2x+2x = 0$.

Collect on each side $5x = 10$.

Divide each side by 5 $x = 2$.

The student should go over the above work very carefully, and find out the reason for adding 2 rather than any other number, for adding $2x$, and for dividing by 5.

57. Verifying the Solution of an Equation. If the work in the last section is correct, when 2 is put instead of x in the equation $3x-2 = 8-2x$, the two sides should be equal. Verify in the following way:

When $x = 2$, $3x-2 = 3.2-2 = 6-2 = 4$.

When $x = 2$, $8-2x = 8-2.2 = 8-4 = 4$.

$\therefore$ when $x = 2$, $3x-2 = 8-2x$; and the solution is correct.

Method of Work. In solving an equation the following points should be carefully observed:

1. Begin by copying the equation in its original form.

2. Never begin a line with the sign =.

3. As far as possible keep the signs of equality in the same vertical line, one under the other.

4. Always explain on the left hand what you are doing.

Examples VIII a.

By simplifying each side of the following equations, find out whether any of them are identities; if they are not, solve them by using the axioms; in each case verify the solution.

1. $2(x-2) = x-3$.
2. $5-2(x-3) = 3x-9-5(x-4)$.
3. $4+12x = x+15$.
4. $-3+5x = 7x+5$.
5. $3x-(x+7) = x-15$.
6. $80-4(x+1) = 6+x$.
7. $13(2x-1) = 5(5x+7)$.
8. $4(9x-3) = 6+5(6x+3)$.
9. $5x+3-4x+7+2x = -2$.
10. $0 = 3x-5-2(x-2)+1-x$.

Does $x = 5$ satisfy any of the following equations?

11. $2x - 3(4 - x) = 7 + x$.

12. $(x - 2)(x - 3) = (x - 4)(x + 1)$.

13. $x^2 - 4x - 2 = 3 + 5x - x^2$.

14. What value of x will make $x - 7 = 0$?

15. What value of x will make $x - 6$ twice as much as $4 - 3x$?

16. What is the value of a if $x = 5$ is a solution of

$$x^2 - 4ax + 15 = 0?$$

17. If $x = -2$, what must be the value of y in order that $3x - 4y = 5$?

18. If $y = -1$ and $z = +2$, what must be the value of x in order that $2x + 4y - 7z = 24$?

19. Is the equation $(y - z)^2 + (z - x)^2 = (x - y)^2$ satisfied by

$$x = 4,\ y = 3,\ z = 2?$$

20. What value of x will make $x - a = a - x$?

58. Transposing. Let us look again at the work in § 56.

The original form of the equation is $3x - 2 = 8 - 2x$.

After two steps the equation becomes $3x + 2x = 8 + 2$.

Comparing these two forms it is seen that the second form can be got from the first by taking the 2 across from the left-hand side to the right-hand side, and at the same time changing its sign, and also taking the $2x$ across from the right-hand side to the left-hand side, and at the same time changing its sign. Very likely the student has already noticed similar results when working the last set of examples. Taking a quantity across from one side to the other is called **transposing.**

Rule for Transposing. Any quantity may be transposed from one side to the other if its sign is changed at the same time.

59. Fractions. If there are fractions on either side of an equation they can be removed by applying axiom 8, § 56.

Example. *Solve* $\dfrac{x}{8} - \dfrac{x-2}{5} + \frac{1}{10}(x - 22) + \dfrac{x-12}{20} = \dfrac{32-x}{40}$.

$$\frac{x}{8}-\frac{x-2}{5}+\tfrac{1}{10}(x-22)+\frac{x-12}{20}=\frac{32-x}{40}.$$

Multiply by 40 the L. C. M. of the denominators

$$\frac{x}{8}\times 40-\frac{x-2}{5}\times 40+40\times\tfrac{1}{10}(x-22)+40\times\frac{x-12}{20}=\frac{32-x}{40}\times 40.$$

Cancel $5x-8(x-2)+4(x-22)+2(x-12)=32-x.$

Remove brackets $5x-8x+16+4x-88+2x-24=32-x.$

Collect on each side $11x-8x+16-112=32-x.$

i.e. $3x-96=32-x.$

Transpose x terms to the left, numbers to the right.

$$3x+x=32+96.$$

Collect $4x=128.$

Divide by 4 $x=32.$

Verification.

If $x=32$, left-hand side $=\frac{32}{8}-\frac{30}{5}+\frac{10}{10}+\frac{20}{20}$

$=4-6+1+1$

$=0;$

and right-hand side $=\frac{0}{40}$

$=0.$

$\therefore$ $x=32$ is the correct solution.

Literal Equations. It sometimes happens that an equation contains not only the letter whose value is to be found, but also other letters. The values of these letters are supposed to be known, and the letters are transposed to the right-hand side like ordinary numbers.

Example. *Solve* $(x-a)+(x-b)-(c-x)=0.$

Remove brackets $x-a+x-b-c+x=0.$

Transpose and collect $3x=a+b+c.$

Divide by 3 $x=\frac{a+b+c}{3}.$

Example. *If* $3y+4x=7$, *find* y *in terms of* x.

Here y is the unknown quantity and x is treated as a known quantity.

$$3y+4x=7.$$

Transpose $3y=7-4x.$

Divide by 3 $y=\frac{7-4x}{3}.$

Examples VIII.

Solve the following equations and verify the solutions:

1. $7x-5x+3x=15$.
2. $5+9x=2x+19$.
3. $32x-7=4x+5+44$.
4. $-2x-5-6x=8x+21-6-6x$.
5. $5x-7+9-4x-3=7-2x+3x-8$.
6. $2x-(x-7)=15$.
7. $3(x+2)-3(x-2)=x+12$.
8. $4(x-3)+5(x-4)+6(x-1)=7$.
9. $4(2-x)-3(2x-3)=7(2+x)-16x-1$.
10. $2(3+2x)+4(6+5x)+3(x+1)+4(x+1)=6$.
11. $(x-4)(x-5)=(x-6)(x-2)$.
12. $(x+5)(x-4)=(x-3)(x+2)-(x-2)-1$.
13. $3\{x-2(x-4)\}=2(x-5)-1$.
14. $2(x-3)(x-4)=2(x-6)(x-1)-(2x-4)$.
15. $3-\{x-2(3x-4)\}=x^2-(x-1)(x+1)+4$.
16. $\frac{x}{12}-\frac{x}{16}=\frac{x}{6}+\frac{7}{12}$.
17. $\frac{1}{2}x-\frac{1}{3}x+7=\frac{5}{6}x-5$.
18. $\frac{x}{8}-\frac{x}{4}+\frac{1}{6}=\frac{x}{8}\quad\frac{1}{12}$.
19. $\frac{3x+1}{3}=\frac{x-1}{4}$.
20. $\frac{x-1}{2}+\frac{x-3}{3}=1$.
21. $5(x-2)-3(3x-4)=17-7x$.
22. $2(x-3)-3(x+1)+2(x-1)+4=0$.
23. $(x+2)(x+3)=(x+4)(x+5)+2$.
24. $(x-1)(x-3)+(x-2)^2=2(x-2)(x-3)$.
25. $\frac{4x-5}{9}-\frac{3x-5}{11}=\frac{2x-5}{4}$.
26. $x(a+2)-6=a(x+6)-2$.
27. What number is that which exceeds its sixth part by 10?
28. If ducks' eggs cost 4*d.* a dozen more than hens' eggs, determine the price of each per dozen, when 7 ducks' eggs and 19 hens' eggs can be bought for two shillings.
29. A man sold a horse for £35 and half as much as it cost him, and by doing so he gained £10. What was the original price of the horse?

30. Find the number the half of which exceeds the difference of its third and fourth parts by ten.

31. A bag contains 129 coins, some of which are sovereigns and the rest shillings; they amount altogether in value to £41. 12*s*. How many coins of each kind are there?

32. A man has been saving annually a quarter of his income. His income is increased by £100, and he now saves a third of his income and finds that he is saving annually £40 more than before. What was his original income?

33. A man performed a journey of 7 miles in $1\frac{1}{4}$ hours, He walked part of the way at the rate of 4 miles an hour, and he rode the rest of the way at the rate of 10 miles an hour. How many miles did he walk?

Solve the following equations, and in each case verify the solution:

34. $\frac{x+1}{2} - \frac{x+3}{5} + 2\frac{1}{2} = 0.$

35. $\frac{1}{2}(9-2x) = 1\frac{1}{2} - \frac{1}{10}(7x-18).$

36. $\frac{2x+4}{3} - 2(x-1) = 7\frac{1}{3} - 2x.$

37. $\frac{3}{2}x - \frac{1}{3}(4-x) = 2\frac{1}{3} - 3(x-2).$

38. $\frac{2x+1}{3} - \frac{x+6}{5} = x-3.$

39. $\frac{3x+1}{7} - \frac{x+8}{5} + \frac{x+1}{3} = 0.$

40. $\frac{-4x+5}{9} + \frac{6x+1}{8} + \frac{1}{12} = 0.$

41. $\frac{2x-1}{3} - \frac{3x-1}{2} + \frac{5x-1}{6} = 0.$

42. $\frac{7x-4}{15} - 2 + \frac{3x+3}{16} = \frac{7x+1}{20}.$

43. $\frac{x}{3} - 8 + \frac{x+40}{5} = 47 - \frac{x}{4}.$

44. $\frac{x-a}{3} - a = \frac{x-b}{5} - 3b.$

45. A poulterer buys a eggs at the rate of b shillings a hundred. One-third of them are broken before he receives them. At what rate per dozen must he sell them so as neither to gain nor lose?

46. A sum of $2a$ pence is to be divided between b boys and c girls, so that for every penny each boy receives each girl is to receive three halfpence. What will be the share of each boy?

47. A farmer bought a certain number of sheep for forty guineas. He lost one-tenth of them, and sold the rest for half a guinea a head more than he had given for them, and thus gained five guineas. Find the number of sheep he bought.

48. If $3x+2y=3\frac{1}{2}$, find the value of y in terms of x, and hence find the value of y when $x=-2\frac{1}{2}$.

49. Find x in terms of y when

$$3x-4(x-2y)=\frac{3x-4y}{3}-2.$$

50. What is the value of a if $x=-3$ satisfies

$$\frac{2-x}{3}+\frac{a+2x}{4}=\frac{x-3}{5}?$$

51. Find the value of A in order that

$$3(x-3)+4(2x-5)-3(x-2)=A(x-1)-15$$

may be an identity.

52. If $\frac{2(x-1)}{5}-\frac{11(x-2)}{\quad}=1$ is satisfied by $x=\frac{1}{2}$, find the missing denominator.

53. Solve $\frac{1}{3x}-\frac{1}{4x}=\frac{1}{24}$.

54. Solve $(x+2\frac{1}{2})(x-1\frac{1}{2})-(x-3)(x-5)=-\frac{3}{4}$.

55. Given $2y=5x-6$, find the value of y when $x=3{\cdot}2$ and when $x=-2{\cdot}4$.

56. Solve $2x-5\{3x-7(4x-9)\}=66-9(x-3)$.

57. Solve $\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}+\frac{1}{4x}=\frac{1}{6}$.

58. A cruiser running at the rate of 16 miles an hour is pursuing an enemy's ship 18 miles off running at the rate of 12 miles an hour. How many miles can the latter run before she is overtaken?

59. A has a dozen more apples than B. If A were to give one-fourth of his apples to B, B would have 6 more than A. How many has A?

60. A train travelling 40 miles an hour takes two hours less in going a certain distance than a train travelling 24 miles an hour. What is the distance?

CHAPTER IX

SIMULTANEOUS EQUATIONS. GRAPHICAL SOLUTIONS

60. If an equation contains two unknown quantities it may be satisfied by many pairs of values. The equation $x+y=50$ is satisfied by $x=1,\ y=49$; $x=8,\ y=47$; $x=-8,\ y=53$; and many other values. Similarly the equation $x-y=7$ has a large number of solutions, and it may be necessary to find what values of x and y will satisfy both equations.

When two or more equations have to be satisfied at the same time, that is, when they are to be satisfied by the same values of the unknown letters, they are said to be **simultaneous.**

Example. *The sum of the lengths of two posts is* 50 *feet, and one is* 8 *feet longer than the other. Find their lengths.*

Let x feet be the length of the longer and y feet be the length of the shorter; then from the question

$$x+y=50,$$
$$x-y=8.$$

Since these equations are to be satisfied by the same values of x and y they are simultaneous.

61. In all methods of solving simultaneous equations the object in view is to **eliminate** or get rid of one of the unknown letters; the resulting equation with only one unknown is then solved by the methods explained in the last chapter.

62. Method I.

Obtain two equations in each of which one of the unknowns has the same numerical coefficient.

Example I. *Solve* $2x+3y=8,$ (i)

$$5x-4y=-8.$$ (ii)

Multiply equation i by 4. $8x+12y=32.$ (iii)

Multiply equation ii by 3. $15x-12y=-9.$ (iv)

Add the last two equations

$$23x = 23.$$

$$\therefore \quad x = 1.$$

Substitute 1 instead of x in equation i

$$2 + 3y = 8.$$

Transpose

$$3y = 6,$$

$$y = 2.$$

Answer. $x = 1,\ y = 2.$

Verification. Equation i. $2 \times 1 + 3 \times 2 = 2 + 6$

$$= 8.$$

Equation ii. $5 \times 1 - 4 \times 2 = 5 - 8$

$$= -3.$$

Example II. *Solve* $21x - 19y = 101,$ (i)

$$35x + 23y = 59.$$ (ii)

Multiply equation i by 5. $105x - 95y = 505.$

Multiply equation ii by 3. $105x + 69y = 177.$

Subtract

$$-164y = 328.$$

$$\therefore \quad y = -2.$$

Substitute in equation i

$$21x + 38 = 101.$$

Transpose

$$21x = 63,$$

$$x = 3.$$

Answer. $x = 3,\ y = -2.$

Examples IX a.

Solve (and in each case verify) the following simultaneous equations:

1. $3x + 5y = 11,$
   $6x - 5y = 7.$
2. $6x + 17y = 28,$
   $5x + 13y = 21.$
3. $-2x + 7y = 27,$
   $13x - 4y = 32.$
4. $15x + 21y = 12,$
   $21x - 35y = -112.$
5. $x - 3y = -2,$
   $4x - 9y = 7.$
6. $5x + 3y = 120,$
   $10x = 9y + 90.$
7. $9x + 24y = 26,$
   $17x - 3y = 33.$
8. $5x - 7y = 22a,$
   $4x + 23y = -11a.$
9. $ax - by = 3ab,$
   $ax + by = 5ab.$
10. $\frac{x}{4} + \frac{y}{3} = 8,$
    $\frac{x}{3} - \frac{y}{5} = 1.$
11. $2x + y + 12 = 0,$
    $x - 6y + 19 = 0.$
12. $x + 3y = 4x + y = 2\frac{1}{2}.$

63. *Solve* $7x-6y=33$, (i)

$3x+11y=-13$. (ii)

Method II.

From equation i $7x=33+6y$.

$$\therefore \quad x=\frac{33+6y}{7}. \qquad \text{(iii)}$$

From equation ii $3x=-13-11y$.

$$\therefore \quad x=-\frac{13+11y}{2}.$$

But since x and y have the same values in both equations these values of x must be the same.

Hence $$\frac{33+6y}{7}=-\frac{13+11y}{3}.$$

Multiply by 21 $99+18y=-91-77y$.

Transpose $95y=-190$,

$y=-2$.

Substitute in iii $$x=\frac{33-12}{7}$$

$=3$.

Answer. $x=3$, $y=-2$.

Method III.

From equation i $7x=33+6y$.

$$\therefore \quad x=\frac{33+6y}{7}. \qquad \text{(iii)}$$

Substitute for x in equation ii

$$\frac{3(33+6y)}{7}+11y=-13.$$

Multiply by 7 $99+18y+77y=-91$.

Transpose $95y=-190$.

$\therefore \quad y=-2$.

Substitute in iii $$x=\frac{33-12}{7}=3.$$

Examples IX b.

Use Methods II and III to solve the following simultaneous equations, and in each case verify:

1. $x+7y=26$,

$\frac{x}{5}+\frac{y}{3}=2$.

2. $7x-5y+6=13y-5x-6$,

$2y-x=2$.

3. $\frac{x}{3}+\frac{y}{7}=1$,
$10x-4y=1$.

4. $\frac{x}{5}+\frac{y}{8}=-1$,
$7x+2y=3$.

5. $3x=4y-7$,
$3y=4x-7$.

6. $x=\frac{5y-7}{3}+2$,
$y=\frac{3x-1}{4}-2$.

7. $x=a(y-b)+b=b(y-b)+a$.

8. $\frac{x+1}{2}=8-\frac{y-1}{3}$.
$\frac{y+1}{2}=9-\frac{x-1}{3}$.

9. $\frac{x}{2}+y=9, \frac{x}{6}=y-1$.

10. $x-3=\frac{y-5}{6}$,
$3y-2=4(x-5)$.

Graphical Method of solving Simultaneous Equations.

64. It has been shown in Chapter VI that if x and y satisfy an equation, the point whose co-ordinates are x and y will lie on a definite curve. This curve is called the **graph** of the equation.

If a point lies on two curves, that is, if it is the point where two curves meet, its co-ordinates will satisfy two equations. And, if x and y satisfy two equations, the corresponding point will be at the intersection of the graphs of the equations.

Example. *Solve* $2x+3y=8$, (i)
$5x-4y=-3$. (ii)

Both equations are of the first degree, therefore the graphs will be straight lines (see § 49).

From equation i $y=\frac{8-2x}{3}$;

if $x=$	1,	7,
$y=$	2,	-2.

From equation ii $y=\frac{5x+3}{4}$;

if $x=$	5,	9,
$y=$	7,	12.

These graphs are shown in Fig. 23. It is seen that they intersect at the point whose co-ordinates are $x = 1$, $y = 2$. Hence these values must satisfy both equations.

Answer. $x = 1$, $y = 2$.

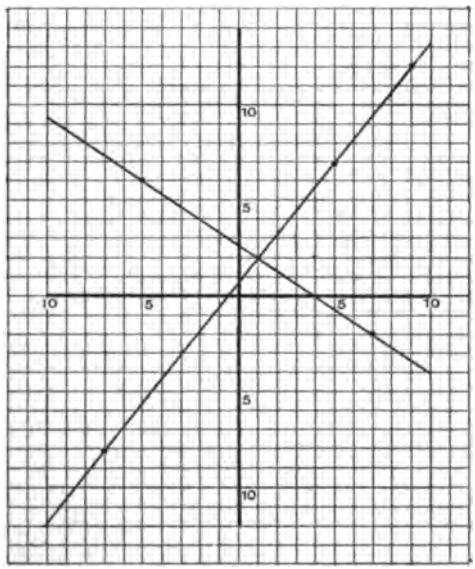

FIG. 23. Scale : 1 side = 1 unit.

65. The graphical method is especially useful when it is not required to have an absolutely correct answer; for practical purposes an answer correct to two significant figures is often quite near enough.

Example. *Solve* $8x + 11y = 17$, (i)

$18x - 19y = 23$. (ii)

The equations are of the first degree, therefore both graphs are straight lines.

From equation i $\quad x = \dfrac{17 - 11y}{8}$;

if $y =$	-5,		11		19,
$x =$	9,		-13		-24.

From equation ii $x = \dfrac{23+19y}{13}$;

if $y=0$,	10		− 10,
then $23y+19=$	23	213	−167,
$x=$	1·77	16·38	− 12·85.

It is better to find whole number values; but with awkward numbers this might take a good deal of time. The answer is wanted only approximately, so that we may be content with approximate values for the plotted points.

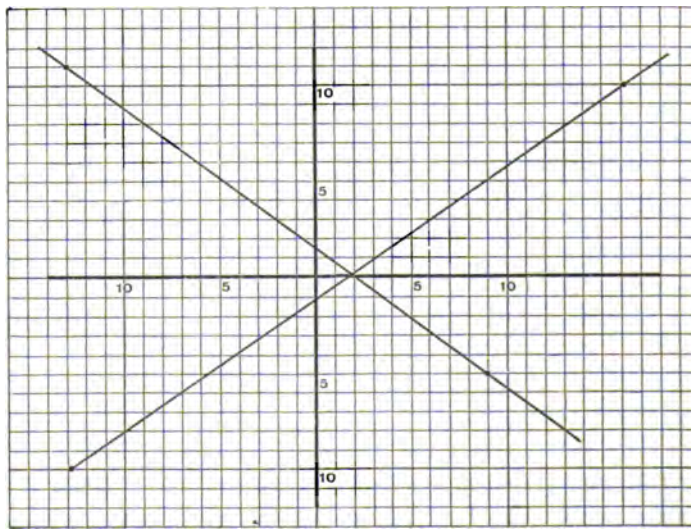

Fig. 24. Scale: 1 side = 1 unit.

It is seen from Fig. 24 that the graphs intersect at a point where $x = 2$ almost exactly and $y =$ rather more than ·1.

Answer. $x = 2{\cdot}0$, $y = {\cdot}1$.

The correct answers are $x = 2 - \frac{14}{295}$, $y = \frac{37}{295}$

$= 2 - {\cdot}04$, $y = {\cdot}12$.

Some care is necessary in estimating the decimal place. The side of the square can by eye be divided into four equal

parts. If the length is just over the half it is reckoned as ·6, if just less as ·4; if just over the first quarter ·8, if just less ·2, &c.

66. It should be noted that the graphical method may be applied when the graphs are not straight lines, the solution always being given by the co-ordinates of the point of intersection. If there is more than one point of intersection, there is more than one solution; if there are no points of intersection, there are no solutions.

Examples IX c.

Solve by graphs (and verify by solving by one of the other methods or by substitution):

1. $3x-5y=-34,$ $2x+8y=34.$
2. $7x-3y=27,$ $4x+5y=2.$
3. $2x+y=8,$ $x+y=1.$
4. $9x-2y=-34,$ $3x+11y=82.$
5. $2-3x=19y,$ $4x-9y=37.$

In the following examples find the answers correct to the first decimal place:

6. $3x-4y=5,$ $-5x+6y=3.$
7. $2x+11y=25,$ $7x-4y=12.$
8. $8x-y=9,$ $13x+17y=2.$
9. $2x+6y=15,$ $3x-5y=13.$
10. $13x-25y=14,$ $11x+19y=18.$
11. $3x-4y=10,$ $x^2+y=1.$
12. $y=x^2-2x,$ $2y=1.$

67. Further hints on solving simultaneous equations.

1. Either or both of the equations may have to be simplified by clearing of fractions, removing brackets, &c.

2. If $A=B=C$, then $A=B$, $A=C$, $B=C$.

Simultaneous equations are sometimes set in this form.

Example. *Solve* $x-7y-2=3x-4y+6=3-2x.$

The first step is to write down two separate equations, e.g.

$$x-7y-2=3-2x, \quad \text{(i)}$$
$$3x-4y+6=3-2x. \quad \text{(ii)}$$

Simplify each of them:

$$3x-7y=5, \quad \text{(iii)}$$
$$5x-4y=-3. \quad \text{(iv)}$$

Finish by either of Methods I, II, III.

3. The equations may contain $\frac{1}{x}$ and $\frac{1}{y}$ instead of x and y.

Example. *Solve* $\frac{3}{x}+\frac{4}{y}=18, \quad \text{(i)}$

$$\frac{5}{x}-\frac{3}{y}=1. \quad \text{(ii)}$$

Treat the numerators exactly as the coefficients are treated in Method I.

Equation i × 3	$\frac{9}{x}+\frac{12}{y}=54.$
Equation ii × 4	$\frac{20}{x}-\frac{12}{y}=4.$
Add	$\frac{29}{x}=58.$
Multiply by x	$29=58x.$
	$\therefore \quad x=\frac{29}{58}=\frac{1}{2}.$
Substitute in i	$6+\frac{4}{y}=18$
	$\frac{4}{y}=12.$
	$\therefore \quad y=\frac{1}{3}.$

4. If the coefficients are letters or contain letters, the methods adopted are exactly the same as with numerical coefficients.

Example. *Solve* $px+qy=r, \quad \text{(i)}$

$$qx-py=o. \quad \text{(ii)}$$

Equation i × p	$p^2x+pqy=pr.$
Equation ii × q	$q^2x-pqy=o.$
Add	$(p^2+q^2)x=pr.$
	$\therefore \quad x=\frac{pr}{p^2+q^2}.$

We could find y by substituting for x in the simpler of the original equations, in this case equation ii. But with letter coefficients it is often easier to begin again and eliminate x.

Equation i $\times q$	$pqx+ q^2y = qr.$	
Equation ii $\times p$	$pqx- p^2y = o.$	
Subtract	$(p^2+q^2)\,y = qr$	
	$y = \frac{qr}{p^2+q^2}.$	

Answer. $x = \frac{pr}{p^2+q^2}$, $y = \frac{qr}{p^2+q^2}$.

68. We may have more than two unknowns. In order to obtain definite solutions we must have as many equations as unknowns.

Example. *Solve* $2x-3y+4z=20,$ (i)

$3x+4y-5z=-20,$ (ii)

$4x-5y-6z=-4.$ (iii)

Eliminate one of the letters from two of the equations:

i $\times 5$	$10x-15y+20z = 100.$	
ii $\times 4$	$12x+16y-20z = -80.$	
Add	$22x+ y = 20.$	(iv)

Eliminate the same letter from another pair of the equations:

i $\times 3$	$6x- 9y+12z = 60.$	
iii $\times 2$	$8x-10y-12z = -8.$	
Add	$14x-19y = 52.$	(v)

iv and v are a pair of simultaneous equations and can be solved by one of the methods already shown.

From iv	$y = 20-22x.$	(vi)
Substitute in v	$14x-380+418x = 52.$	
Transpose and collect	$432x = 432.$	
	$\therefore\ x = 1.$	
Substitute in vi	$y = 20-22 = -2$	

Substitute in any of the original equations.

Substitute in i	$2+6+4z = 20.$	
Transpose and collect	$4z = 12,$	
	$z = 3.$	

Answer. $x = 1,\ y = -2,\ z = 3.$

Examples IX d.

Solve:

1. $x-\frac{2}{5}y=3\frac{1}{2}$, $\frac{4}{9}x+y=-4\frac{1}{3}$.

2. $3(x-2)-2(y+3)=1$, $2(x-3)+3(y+2)=0$.

3. $\frac{3}{x}-\frac{4}{y}=2$, $\frac{4}{x}+\frac{5}{y}=13$.

4. $\frac{4}{x}-\frac{5}{y}=\frac{2}{a}$, $\frac{3}{x}+\frac{6}{y}=\frac{21}{a}$.

5. $ax+by=a^2-b^2$, $2bx-3ay=5ab$.

6. $17x+19y=20$, $19x+17y=16$.

7. $4x-3y-z+1=0$, $2x+2y-z-18=0$, $x+2y+z-11=0$.

8. $\frac{6x-2y}{11}-1=\frac{2x-y}{5}$. $\frac{2x+3y}{3}-3=\frac{5x-2y}{2}$.

9. $\frac{3}{x}-\frac{4}{y}=\frac{5}{6}$, $\frac{5}{x}+\frac{6}{y}=3\frac{1}{2}$.

10. $5x-\frac{5y+2}{4}=32$, $3y-\frac{x+2}{3}=9$.

11. $ax-by+c=0$, $b(x-y)=a-b$.

12. $px+qy=qx+py=r$.

13. $3x-4y+7z=7$, $2x-5y-6z=7$, $8x+3y-10z=5$.

14. $2x-4y=2$, $3y-5z=13$, $x+y+z=2$.

15. $x+y-z=a$, $x-y+z=b$, $-x+y+z=c$.

16. $ax+by=ab$, $\frac{x}{a}+\frac{y}{b}=\frac{1}{ab}$.

17. $x+y=1$, $ax+by=a+b$.

18. $\frac{9}{x}-\frac{4}{y}=8$, $\frac{13}{x}+\frac{7}{y}=101$.

19. $\frac{x-a}{2}+\frac{y-b}{3}=a$, $\frac{x-b}{3}+\frac{y-a}{2}=b$.

20. $\frac{x-y}{3}+\frac{x+y}{5}=4$, $\frac{x-y}{5}-\frac{x+y}{9}=1\frac{11}{45}$.

21. $x+2y+z+7=0$,
$2x+y=z+1$,
$3x+2z=y+2$.

22. $\frac{2x-y}{3}=\frac{x+y+1}{4}=\frac{2y-x}{5}$.

23. $8x+7y=2$, $392x-336y=1$.

24. $4x+5y+z=6$,
$x+7y+2z=10$,
$5x-3y-6z=16$.

25. $3x-2y-z=8$,
$x+3y-2z=4$,
$7x-y-4z=20$.

26. $2x+y-3z=9$,
$3x-2y+z=3$,
$13x-4y-3z=19$.

Examples IX.

1. The sum of two numbers is 176 and their difference 48; find the numbers.

2. Three times one number exceeds another by 17; four times the second number exceeds twice the first by two. What are the numbers?

3. If $3x+4y=7=y-x$, find the value of $4x+3y$.

4. Solve $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}=1+\frac{y}{c}$.

5. Solve $\frac{3(x+1)}{7}+\frac{4}{y}=-5$,
$\frac{4(x-1)}{5}-\frac{5}{y}=14$.

6. A sum of £5 is divided between two men, so that one receives eight times as many pence as the other receives shillings. What sum does each receive?

7. *A* has twice as much money as *B* to start with, and saves three times as much as *B* every week. At the end of 20 weeks *A* has £13 and *B* has £6. How much does each save per week?

8. If 5 lb. of tea and 7 lb. of coffee cost £1. 3*s*. 4*d*., and 8 lb. of tea and 3 lb. of coffee cost £1. 3*s*. 8*d*., find the cost of tea and coffee per lb.

9. A page of print contains 1300 words if in large type and 1850 if in small type. If an article of 18625 words occupies exactly 12 pages, how many pages are in small print?

10. The graph of the equation $ax+by=1$ passes through the two points $(3, -2)$ and $(-1, 2)$. Find the values of a and b.

11. What must be the values of m and c if the graph of $y = mx + c$ passes through the two points $(4, -2)$ and $(-3, 2)$?

12. Find the equation of a straight line which cuts the axis of x where $x = 3$ and also passes through the point $(4, 5)$.

13. Find the values of a, b, and c in order that the graph of $y = ax^2 + bx + c$ may pass through three points $(-1, 2)$, $(3, 4)$, $(2, -2)$.

14. The age of a father is twice the sum of the ages of his two sons, the elder of whom is twice as old as the younger. Nine years hence the father will be three times as old as his younger son. Find their ages.

15. A man buys two horses for £90. By selling one for four-fifths of its cost price and the other for five-fourths of its cost price he makes a profit of £6 on the whole transaction. Find the cost price of each horse.

16. If 3 be taken both from the numerator and from the denominator of a certain fraction, the resulting fraction is equal to $\frac{2}{3}$, but if 3 be taken from the numerator and added to the denominator of the same fraction, the resulting fraction is equal to $\frac{1}{3}$. Find the original fraction.

CHAPTER X

MULTIPLICATION

69. It has already been shown that $xy = yx$; it follows that, when there are more than two factors, the multiplications may be performed in any order, e. g.

$$abcd = badc = bdac.$$

70. Suppose it is required to multiply together two different powers of the same quantity, e. g. to find the product of x^3 and x^5.

$$x^3 = 1 \times x \times x \times x,$$
$$x^5 = 1 \times x \times x \times x \times x \times x.$$
$$\therefore\ x^3 \times x^5 = 1 \times x \times x \times x \times 1 \times x \times x \times x \times x \times x$$
$$= 1 \times 1 \times x \times x \times x \times x \times x \times x \times x \times x \quad \text{by § 69}$$
$$= 1 \times x \times x \times x \times x \times x \times x \times x \times x$$
$$= x^8.$$

Other cases could be done in the same way, and it is seen that when two powers of x are multiplied together, the product is a power of x whose index is the sum of the original indices.

The same rule applies for a number or for any letter or for any algebraical quantity.

$5^3 \times 5^4 = 5^7$, $a^2 \times a^3 = a^5$, $(x+a)^2 \times (x+a)^4 = (x+a)^6$.

71. Each of the two powers may be preceded by a coefficient.

Example I. *To find the product of* $3x^3$ *and* $7x^5$.

$$\begin{aligned} 3x^3 \times 7x^5 &= 3 \times x^3 \times 7 \times x^5 \\ &= 3 \times 7 \times x^3 \times x^5 \quad \text{by § 69} \\ &= 21x^8 \quad \text{by § 70.} \end{aligned}$$

The same reasoning can be applied to all cases.

Example II. *To find the product*

$$-2a^2bc^3 \times 3ab^2c^2 \times 4a^3bc^4.$$

$$\begin{aligned} \text{Product} &= -2a^2bc^3 \times 3ab^2c^2 \times 4a^3bc^4 \\ &= -2 \times 3 \times 4 \times a^2 \times a \times a^3 \times b \times b^2 \times b \times c^3 \times c^2 \times c^4 \\ &= -24a^6b^4c^9. \end{aligned}$$

Examples X a.

Find the following products:

1. $2x \times 3x^3$.
2. $2a \times 3x^3$.
3. $2abc^2 \times 3ab^2c^4$.
4. $5x^2 \times 3x^4$.
5. $7a^2 \times 4a^3$.
6. $4ax^2 \times 6a^2xy$.
7. $2y^3 \times 7y^5$.
8. $9z^2 \times 3xyz$.
9. $3xy^2 \times 5x^2y$.
10. $3xy \times 2yz \times xz$.
11. $3xy^2 \times 5x^2y^4$.
12. $-2ab \times abc^2$.
13. $4x^{15} \times 15x^4$.
14. $3p^2q \times 7pq^3$.
15. $3a \times 4b \times 5c$.
16. $7x^2y \times 8xy^4$.
17. $x^5 \times x^{19} \times x^{15}$.
18. $3a^2b^5 \times 4a^3b^4 \times (-b^2)$.
19. $8a^2 \times 9b^2 \times 6c^2$.
20. $3abc \times 4xyz$.
21. $21a^2 \times 2b^{21} \times 3a^2b$.

What must $2a^2x^3$ be multiplied by to give the following products?

22. $6a^3x^5$.
23. $-8a^4x^7$.
24. $4a^4x^9$.

72. The simpler cases of Multiplication have already been dealt with in the chapter on Brackets.

Example. *Multiply* $3x^3-2x^2+x-4$ *by* $2x$.

$$\begin{aligned}\text{Product} &= 2x(3x^3-2x^2+x-4)\\ &= 6x^4-4x^3+2x^2-8x,\end{aligned}$$

since the quantity outside the bracket multiplies every term inside. The work can also be set down as follows very much in the same way as in Arithmetic.

$3x^3-2x^2+x-4$		4 3 5 4	
$2x$		2	
$6x^4-4x^3+2x^2-4x$	Product.	8 7 0 8	

It is usual to place the multiplier immediately under the left-hand term in Algebra, although it does not really matter where it is placed. There is a good deal to be said for arranging the work so that all like powers come in the same column, as follows:—

$3x^3-2x^2+\ x-4$	
$2x$	
$6x^4-4x^3+2x^2-4x$	Product.

73. Mental Multiplication. It is shown in § 39 that $(x+a)(x+b)=x^2+(a+b)x+ab$; it should be noticed that the coefficient of x in the product is the algebraic sum of the two constants in the factors (either a or b, or both, may be negative) and the constant in the product is the product of the original constants. Hence all similar multiplications can be done mentally.

$$\begin{aligned}(x+3)(x+5) &= x^2+(5+3)x+5\times 3\\ &= x^2+8x+15,\\ (x-3)(x+5) &= x^2+(-3+5)x+(-3)\times 5\\ &= x^2+2x-15.\end{aligned}$$

The process is easily reversed.

Example. *What two numbers multiplied together give a product:*

(i) $x^2-8x+12$, (ii) $x^2-8x-20$?

(i) $x^2-8x+12 = x^2-(6+2)x+(6\times 2)$
$= (x-6)(x-2).$

(ii) $x^2-8x-20 = x^2+(-10+2)x+(-10)\times(+2)$
$= (x-10)(x+2).$

Examples X b.

Find the product of:

1. x^2-2x-1 and $3x$.
2. $3y^3-2y^2+4y-1$ and $2y^2$.
3. $3a^3-3a^2-2$ and $4a^2$.
4. $x^4-2x^3+4x^2-3x+2$ and $4x^3$.
5. $5a^4-4a^3+a-2$ and $-3a^3$.
6. $x-x^3+2x^2-4$ and $3a$.
7. $9x^4-3x^3+2-x$ and $4x^4$.
8. $k^4-k^3+k^2-k+1$ by $-k^2$.

Find mentally the products of:

9. $x+5$ and $x+3$.
10. $x-5$ and $x+3$.
11. $x-2$ and $x+7$.
12. $y-3$ and $y-4$.
13. $x-4$ and $x+6$.
14. $x-6$ and $x-8$.
15. $x+5$ and $x-5$.
16. $x+5$ and $x-9$.
17. $x+7$ and $x-7$.
18. $y+3$ and $y-3$.
19. $a+2$ and $a+2$.
20. $a+2$ and $a-2$.
21. $x+12$ and $x-10$.
22. $x-14$ and $x+7$.
23. $x-6$ and $x+6$.
24. $2x-5$ and $2x+5$.
25. $y-8$ and $y-2$.
26. $a-15$ and $a-11$.
27. $z-3$ and $z-7$.
28. $x-1$ and $x+50$.
29. $x-5$ and $x+8$.
30. $x+6$ and $x+9$.

What quantities multiplied together give the following products?

31. $x^2+8x+12$.
32. $x^2+7x+12$.
33. $x^2-8x+12$.
34. $x^2-7x+12$.
35. $x^2-10x+9$.
36. $x^2+7x+10$.
37. x^2-3x-4.
38. $x^2-5x-14$.
39. x^2+3x-4.
40. $x^2+5x-14$.
41. x^2-4.
42. x^2-9.

74. Example. *Multiply* $3x^3-2x^2+x-4$ *by* $2x-3$.

$$\begin{aligned}\text{Product} &= (2x-3)(3x^3-2x^2+x-4)\\ &= 2x(3x^3-2x^2+x-4) - 3(3x^3-2x^2+x-4)\\ &= 6x^4-4x^3+2x^2-8x\\ &\qquad -9x^3+6x^2-3x+12\\ &= 6x^4-13x^3+8x^2-11x+12.\end{aligned}$$

Either of the following arrangements may be used:

$$\begin{array}{l} 3x^3 - 2x^2 + x - 4 \\ 2x - 3 \\ \hline 6x^4 - 4x^3 + 2x^2 - 8x \\ \quad - 9x^3 + 6x^2 - 3x + 12 \\ \hline 6x^4 - 13x^3 + 8x^2 - 11x + 12 \end{array}$$ Product.

$$\begin{array}{l} 3x^3 - 2x^2 + x - 4 \\ 2x - 3 \\ \hline 6x^4 - 4x^3 + 2x^2 - 4x \\ \quad - 9x^3 + 6x^2 - 3x + 12 \\ \hline 6x^4 - 13x^3 + 8x^2 - 7x + 12 \end{array}$$ Product.

Example. *Find the product of* $3 - x^3 + 4x^2 - 3x$ *and* $2x^3 + x - 2x^2 + 3$.

Before multiplying, both expressions must be arranged in descending powers of x, or in ascending powers.

(i) Descending powers.

$$\begin{array}{l} 2x^3 - 2x^2 + x + 3 \\ \quad - x^3 + 4x^2 - 3x + 3 \\ \hline -2x^6 + 2x^5 - x^4 - 3x^3 \\ \quad + 8x^5 - 8x^4 + 4x^3 + 12x^2 \\ \quad\quad - 6x^4 + 6x^3 - 3x^2 - 9x \\ \quad\quad\quad + 6x^3 - 6x^2 + 3x + 9 \\ \hline -2x^6 + 10x^5 - 15x^4 + 13x^3 + 3x^2 - 6x + 9 \end{array}$$ Product.

(ii) Ascending powers.

$$\begin{array}{l} 3 - 3x + 4x^2 - x^3 \\ 3 + x - 2x^2 + 2x^3 \\ \hline 9 - 9x + 12x^2 - 3x^3 \\ \quad + 3x - 3x^2 + 4x^3 - x^4 \\ \quad\quad - 6x^2 + 6x^3 - 8x^4 + 2x^5 \\ \quad\quad\quad + 6x^3 - 6x^4 + 8x^5 - 2x^6 \\ \hline 9 - 6x + 8x^2 + 13x^3 - 15x^4 + 10x^5 - 2x^6 \end{array}$$ Product.

Multiplication sums can be checked by giving x a numerical value and working the resulting arithmetical sum. In the last example, put $x = 1$.

Then $8-3x+4x^2-x^3 = 3-3+4-1 = 3$,

$3+x-2x^2+2x^3 = 3+1-2+2 = 4$;

and the product found above $= 9-6+8+13-15+10-2 = 12 = 4 \times 3$.

The results of the following multiplications are so important that they should be learnt by heart.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array} \qquad \begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad -b^3 \end{array}$$

$$\begin{array}{l} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad +a^2b-ab^2+b^3 \\ \hline a^3 \qquad\qquad +b^3 \end{array}$$

The formulae to be recollected are therefore:

$$(a+b)(a-b) = a^2-b^2,$$
$$(a-b)(a^2+ab+b^2) = a^3-b^3,$$
$$(a+b)(a^2-ab+b^2) = a^3+b^3.$$

There is no formula giving the factors of a^2+b^2.

75. Sometimes only one term is wanted in a product, e.g. find the term containing x^2 in the product of $3x^3-2x^2+x-4$ and $2x^3-3x^2+4x+5$.

Here it would be labour wasted to work the sum in full. Starting with $2x^3$, it is seen that no term containing x^2 can be obtained from the multiplication by $2x^3$; going to $-3x^2$ it is seen that, in order to obtain a term involving x^2, $-3x^2$ must be multiplied by a mere number, in this case -4. Proceeding in

a similar manner with the other terms, the whole work would appear thus:—

Term involving $x^2 = +12x^2+4x^2-10x^2$
$= 6x^2$.

Examples X c.

Multiply together:

1. $3x^3-2x^2+x-2$ and $x-2$.
2. $4x^4-3x^3+2x^2+x-2$ and $3x-4$.
3. $5x^3-2x^2+3x-1$ and $2x-5$.
4. $3x^3-3x^2+2x-5$ and x^2-2x-2.
5. $7x^3-3x+1$ and $3x^2-3x+4$.
6. $3x^3-2x^2y+3xy^2-4y^3$ and $x-2y$.
7. $2a^3-2a^2+3a-4$ and $2a^2-3a+3$.
8. $4a^4-6a^3-2a+1$ and $3a^2-2a-4$.
9. $a-2a^3+3a^2-5$ and $2-a+a^2$.
10. $3ab+2bc+3ac$ by $2a+3b-4c$.

Find the term involving x^3 in the products of:

11. $1+2x+3x^2+4x^3+5x^4$ and $1-x+x^2-x^3+x^4$.
12. $1-3x+5x^2-7x^3+9x^4$ and $1-2x+3x^2-4x^3+5x^4$.
13. $2-4x+6x^2+8x^3+10x^4$ and $3-9x+27x^2+81x^3$.
14. $1-ax+a^2x^2-a^3x^3+a^4x^4$ and $2-3bx+4b^2x^2-6b^3x^3$.
15. x^4-2x^3+3x-4 and $x^5-43x^4+x^3-2x+3$.

Find mentally the products of:

16. $x-2y$ and $x+2y$.
17. $x+2y$ and $x^2-2xy+4y^2$.
18. $p-3q$ and $p^2+3pq+9q^2$.
19. $2a-3b$ and $2a+3b$.
20. $2a+3b$ and $4a^2-6ab+9b^2$.
21. $3x-5y$ and $9x^2+15xy+25y^2$.
22. $p-4$ and $p^2+4p+16$.
23. Multiply $3a^3-4a^2+2a-7$ by $2a^2+3a-1$.
24. Find the product of $3-2x-3x^2$ and $2-3x^2-5x^3$.
25. What must be multiplied by $x-3$ to give a product $x^2-8x+15$?
26. What must be multiplied by $x-3$ to give a product x^3-27?
27. Multiply ax^2+bx+c by bx^2-cx-a.
28. Add the product $(a+b+1)(a-b+2)$ to the product $(a+b-1)(a-b-2)$

and verify the result when $a=4$, $b=1$.

29. Multiply together (x^2+x+1), x^2-x+1, and $x+1$.

30. Find the product of

$$a(1+b)-1 \text{ and } a^2(1-b+b^2)+a(1+b)+1.$$

Multiply together:

31. $\frac{1}{2}a-\frac{1}{3}b$ and $\frac{1}{2}a+\frac{1}{3}b$.

32. $\frac{1}{2}a+\frac{2}{3}b-\frac{1}{5}c$ and $\frac{1}{3}a-\frac{1}{2}b$.

33. $\frac{2}{3}x^2-\frac{1}{4}x+\frac{2}{5}$ and $\frac{1}{3}x^2-\frac{2}{5}x+\frac{1}{4}$.

34. $2a^2-\frac{1}{3}a+\frac{1}{7}$ and $\frac{1}{2}a^2-\frac{2}{3}a+\frac{1}{2}$.

35. $\frac{a}{3}-\frac{1}{5}$ and $\frac{a^2}{9}+\frac{a}{15}+\frac{1}{25}$.

Examples X.

1. Find the product of the three factors $(x+y)$, $(y+z)$, $(x+z)$. Arrange the answer in descending powers of x.

Test your result by putting $x=-1$, $y=2$, $z=3$ in the factors and in the product.

2. Multiply $4a-3b+5c$ by $-6a^2b^2$.

3. If $P=x^2+2xy+2y^2$, $Q=x^2-2xy+2y^2$, $R=x^4-y^4$, find the value of $PQ-R$.

4. Multiply $5x^2-xy^2-7y^3$ by $x^2-4xy+3y^2$. Verify the result when $x=-2$, $y=1$.

5. What is the product when $(1+x)(1+x^4)$ is multiplied by $(1-x)(1+x^2)$?

6. Multiply $4a^2-2a+1$ by $4a^2+2a+1$.

7. Multiply together $(x+a-b)(x+b-c)(x-a+c)$, collecting in brackets the coefficients of the powers of x.

8. If $2s=a+b+c$, prove that $2s(2s-2a)+(2s-2b)(2s-2c)$ equals $4bc$.

9. Simplify

$$(y-z)(z-x)+(z-x)(x-y)+(x-y)(y-z)+(x+y+z)^2.$$

10. Find the area of a room whose length is x^2 yards x feet 4 inches and breadth x yards 2 feet x^2 inches.

11. What quantities multiplied together give a product x^3+x-6?

12. Multiply $x^3+2ax+2a^2$ by $x^2-2ax+2a^2$.

13. Multiply x^5-x^3-9x+9 by $4x^3+4x^2-x-1$.

14. Multiply $9x^2+3xy+y^2-6x+2y+4$ by $3x-y+2$.

15. What will a oranges, b apples, c pears cost if the price is $(a+2b-c)$ pence each?

16. Multiply $2-3x$ by $8+12x+18x^2+27x^3$.

17. Multiply $5a^3-3a^2b+2ab^2+5b^3$ by $10a^2+6ab-15b^2$.

18. Multiply $\frac{y}{3}-2+\frac{z}{2}$ by $\frac{z}{2}+2+\frac{y}{3}$.

19. Multiply together $1+x-y$, $1-x+y$, $1-x-y$.

20. What is the product of $a+b+c$ and $a^2+b^2+c^2-ab-ac-bc$?

21. What are the factors of x^2-4x-5?

22. Simplify

$$(a-b)(a+b+2c)+(b-c)(b+c+2a)+(c-a)(c+a+2b).$$

23. Multiply together $y+z-2x$, $z+x-2y$, and $x+y-2z$. Put $x=1$, $y=1$, $z=1$ in the result, and hence show how we could tell beforehand that the algebraical sum of the coefficients in the product would be zero.

24. If $x-3$ will divide x^2-5x+c without remainder, what is the value of c?

25. What is the coefficient of x in the product of $(x-2a)$, $(2x+3)$, $(3x-4)$? What value must a have in order that this coefficient shall be zero?

26. Prove that $a^m \times a^n = a^{m+n}$.

27. Show that $(a+c)(a-c)-b(2a-b) \equiv (a-b+c)(a-b-c)$.

Verify when $a=1$, $b=-1$, $c=0$.

28. Prove that $(a+b)c = ac+bc$ when a, b, and c are positive whole numbers.

Simplify $(2a+3b)(2x-3y)-6(a+x)(b-y)$.

29. Multiply together $9x^2+7x+3$ and $3x^2+7x+9$.

Hence find the product of 973 and 379.

30. Multiply $x+y-1$ by $x^2+y^2+1+x+y-xy$.

31. If a and b stand for any numbers, express in words the fact that $(a+b)(a-b) \equiv a^2-b^2$.

32. Simplify by two different methods $(5x-7)^2-(x-5)^2$.

33. If $2b=a+c$, show that $(a-b)^2+2b^2+(b-c)^2 = a^2+c^2$.

34. Simplify

$$(x+1)(x-2)-2(x+2)(x-3)+(x+3)(x-1).$$

35. Simplify

$$(5x+3y)^2-(4x-5y)^2+(3x-4y)[2(x-y)-(5x+2y)].$$

36. Find the sum of the squares of $a-b+c-d$ and $a-b-c+d$.

37. Multiply $\frac{2}{3}x^3-\frac{1}{2}x^2+\frac{1}{3}$ by $\frac{1}{3}x-2$.

38. If $p=b+c-2a$ and $q=c+a-2b$ show that

$$p^2+q^2-2pq = 9(a-b)^2.$$

39. Simplify $7x^3-(3-4ax)-3x(4x+1)(ax+5)$; and re-bracket the result according to descending powers of x.

40. Arrange $a(b^2-c)+b(c^2-a)+c(a^2-b)$ in descending powers of c; and find its value in terms of c when $a = 2b = 3c$.

Solve the following equations:

41. $(3x-5)(4x-7)-2(3x-1)(2x-1) = \frac{5x+3}{4}$.

42. $4(2x+3)(3x+1)-12(x+1)^2 = (4x+5)(3x+1)$.

43. $(x-2)(x-3)-6(x-3)(x-4)+5(x-2)(x-4) = 16$.

44. $(x-1)(x-4)-(x-2)(x-5) = (x+1)^2-x(x-3)-37$.

45. $2(x+2)(4x-5)-(4x+1)(2x-3) = 8x+3$.

46. $(x+3)(x-1)(2x+5) = (x-4)(x+2)(x-5)$
$+(x+10)(x-2)(x+8)$.

47. $(5x-6)(x+1)-(7-4x) = 8(1+x^2)-3(x-2)^2$.

48. $3(x+1)(x+3)-2(x^2-1) = (x-1)^2+3(5x+1)$.

49. Prove the identity
$(a-b+c)(a^2+b^2+c^2+bc-ca+ab) \equiv a^3-b^3+c^3+3abc$.

50. Prove the identity
$(a-b)(x-a-b)+(b-c)(x-b-c)+(c-a)(x-c-a) \equiv 0$.

CHAPTER XI

DIVISION

76. Division is the process that reverses Multiplication, so that 'Divide x by y' is only another way of saying 'What quantity when multiplied by y gives a product x?', or 'What quantity must y be multiplied by to give a product x?'

The quantity to be divided is called the **dividend**, the quantity which divides is called the **divisor**, the result is the **quotient**. Hence dividend = divisor multiplied by quotient.

77. The rule for Division follows at once by reversing the rule for Multiplication.

Rule. **When one power of a quantity is divided by another power of the same quantity, the quotient is a power whose index is obtained by subtracting the**

index of the divisor from the index of the dividend, e. g. $x^7 \div x^3 = x^{7-3} = x^4$.

Example. *Divide* $18\,a^4b^5c^7$ *by* $3\,a^3b^2c^4$.

$$\text{Quotient} = 18\,a^4b^5c^7 \div 3\,a^3b^2c^4$$
$$= \tfrac{18}{3}\,a^{4-3}b^{5-2}c^{7-4}$$
$$= 6\,ab^3c^3.$$

78. A fraction bears the same relation to division that brackets do to multiplication, for $\dfrac{3x^2-5x+7}{2x}$ means $3x^2-5x+7$ divided by $2x$, and each term in the numerator may be divided separately by $2x$.

Example. *Divide* x^3-4x^2+2x *by* $2x$.

$$\text{Quotient} = \frac{8x^3-4x^2+2x}{2x}$$
$$= 4x^2-2x+1.$$

Or a method similar to short division in Arithmetic may be used:

$$2x \;|\; \underline{8x^3-4x^2+2x}$$
$$\underline{\underline{4x^2-2x+1}} \quad \text{Quotient.}$$

79. In Arithmetic there are two ways of expressing the fact that the divisor is not contained exactly in the dividend: $37 \div 5$ gives quotient 7 and remainder 2, or $37 \div 5$ gives quotient $7\frac{2}{5}$. Both methods say that 5 is contained in 37 seven times, and that when the seven fives are taken from 37 there remains 2 still to be divided. Similarly in Algebra: $4x^2-2x+10 \div 2x$ gives quotient $2x-1$ and remainder 10 or

$$(4x^2-2x+10) \div 2x = \frac{4x^2-2x+10}{2x} = 2x-1+\frac{10}{2x}.$$

Examples XI a.

Divide:

1. $6a^2b^4$ by $3ab^2$.
2. $-9a^3b^3c^3$ by $3abc$.
3. $21x^3y^4z^3$ by $-7xy^2z$.
4. $7x^5y^2z^3$ by $-x^2yz$.

5. $-36a^4b^3x^2y$ by $-4a^2bx$.
6. $13a^3x^4y^2$ by $13ax^3y^3$.
7. $14a^3x^5y^7$ by $-2a^2xy^2$.
8. $-15a^4b^3c^3$ by $-5a^3b^2c$.
9. $12abc$ by $6abc$.
10. $7a^4b^3c^3$ by $3a^2bc^3$.

Simplify:

11. $\frac{6x^3-4x^2+2x}{2x}$.
12. $6x^3\times 4x^2\div 2x$.
13. $\frac{9a^4+27a^3-6a^2+3a}{3a}$.
14. $\frac{6y^4+3y^3+9y^2+3y^5}{-3y^2}$.
15. $\frac{6a^2bc^3-12a^3b^4c^5+18a^4b^3c^4}{3a^2bc^2}$.
16. $3x^3\times 2y^2z^3\div 6xyz$.
17. $\frac{1}{2x}(4x^3-6x^2+8x)$.
18. $-\frac{3x^3+3x-6x^2}{x}$.
19. $\frac{x^4-6x^3y+12x^2y^2-3xy^3}{x}$.
20. $\frac{7x^4-14x^3-21x^2-7}{7x}$.

Find the quotient and remainder when

21. $3a^3b^3-6a^2b^2+5ab$ is divided by $3a^2b^2$.
22. $6x^3y^4+4x^2y^3+8xy^2$ is divided by $2xy^2$.
23. $4a^4-6a^3+8a^2-10a+3$ is divided by $2a^2$.
24. $36x^3-24x^2y+16xy^2-y^3$ is divided by $8x^3$.
25. $7x^4y^3-14x^3y^3+35x^2y^4-56xy^5+13y^6$ is divided by $7xy$.

80. It was shown in the previous chapter that such multiplication sums as $(x-4)\times(x+2)$ can be done mentally; the product being $x^2+(-4+2)x+(-4)\times(2)$, i. e.

$$x^2-2x-8.$$

Hence reversing the process

$$(x^2-2x-8)\div(x-4)=x+2.$$

In working division by this mental method, we must see whether the result obtained by subtracting the coefficient of x in the divisor from the coefficient of x in the dividend is the same as the result obtained by dividing the constant in the dividend by the constant in the divisor. If these results are not the same, the division is not exact and there is a remainder.

Example. *Divide* $x^2-8x-14$ *by* $x-10$.

[Mental work

$-8-(-10)=-8+10=+2$; $(-10)\times(+2)=-20$.

$\therefore$ there is a remainder.]

$$x^2-8x-14 = x^2-8x-20+6$$
$$= (x-10)(x+2)+6;$$

$\therefore$ when $x^2-8x-14$ is divided by $x-10$, the quotient is $x+2$ and the remainder is 6.

Or $$\frac{x^2-8x-14}{x-10} = x+2+\frac{6}{x-10}.$$

Examples XI b.

Find the quotient mentally in the following cases:

1. $(x^2-7x+12)\div(x-3)$.
2. $(x^2+7x-18)\div(x+9)$.
3. $(x^2-4x-12)\div(x-6)$.
4. $(y^2+9y+20)\div(y+4)$.
5. $(y^2-16y+48)\div(y-12)$.

Is there any remainder in the following divisions?

6. $(p^2-4p-7)\div(p-5)$.
7. $(a^2+3a-6)\div(a-3)$.
8. $(x^2-3x+4)\div(x-4)$.
9. $(x^2+2x-8)\div(x+4)$.
10. $(y^2-3y+6)\div(y-3)$.

Find the remainder when

11. (a^2-6a+4) is divided by $a-8$; and put $a=8$ in the dividend.
12. a^2-5a-8 is divided by $a-7$; and put $a=7$ in the dividend.
13. x^2+3x-1 is divided by $x+4$; and put $x=-4$ in the dividend.
14. $x^2-2x+15$ is divided by $x+6$; and put $x=-6$ in the dividend.
15. y^2-y-1 is divided by $y-4$; and put $y=4$ in the dividend.

Find mentally the values of:

16. $\frac{a^2-4}{a-2}$.
17. $\frac{a^3-8}{a-2}$.
18. $\frac{x^2-a^2}{x+a}$.
19. $\frac{y^2-16}{y+4}$.
20. $\frac{p^2-1}{p+1}$.
21. $\frac{p^3+1}{p^2-p+1}$.
22. $\frac{x^2-2x+1}{x-1}$.
23. $\frac{x^2-4x+4}{x-2}$.
24. $\frac{y^3+a^3}{y+a}$.
25. $\frac{x^4-a^4}{x^2+a^2}$.

81. Division can be done in Algebra by a method exactly similar to Long Division in Arithmetic.

Example.

Divide $x^5+4x^4-2x^3+x^2-4x+6$ *by* x^2-3x+5.

$x^3+7x^2+14x+8$ Quotient.

$$
\begin{array}{r}
x^2-3x+5\,)\,x^5+4x^4-2x^3+x^2-4x+6 \\
x^5-3x^4+5x^3 \\
\hline
7x^4-\ 7x^3+\ x^2 \\
7x^4-21x^3+35x^2 \\
\hline
+14x^3-34x^2-\ 4x \\
+14x^3-42x^2+70x \\
\hline
+\ 8x^2-74x+\ 6 \\
+\ 8x^2-24x+40 \\
\hline
-50x-34 \quad \text{Remainder.}
\end{array}
$$

Note. Care must be taken that the divisor and dividend are both arranged in ascending powers of x, or both in descending powers. If any power does not occur, a gap should be left where that power would have come. The division must be continued as long as possible, that is, so long as the first term of the divisor will divide the first term of the remainder. Hence the greatest index in the final remainder is always less than that in the divisor.

82. Horner's Method of Long Division. In the work of the last section it is seen that the first term in any remainder is obtained by a number of subtractions, e. g. to obtain $14x^3$, $5x^3$ is subtracted from $-2x^3$, and $-21x^3$ is subtracted from the result. For each subtraction a mental change of sign is made. Horner's method has the following advantages:

(i) The continual mental change of sign is avoided by actually changing the signs at the beginning of the work.

(ii) The work is so arranged that, instead of several subtractions, only one algebraic addition is necessary.

The successive steps of Horner's method should be carefully studied; with a little practice the method will be

found to be not only neater and shorter, but also easier than the method of § 51.

Example.

Divide $x^5+4x^4-2x^3+x^2-4x+6$ *by* x^2-3x+5.

Step 1. Change the sign of all the terms in the divisor except the first and write them down in a vertical column.

Step 2. Obtain the first term in the quotient by dividing the first term in the dividend by the first term in the divisor and place the quotient under the first term of the dividend. The first column is now finished as shown below.

Step 3. Multiply the terms in the divisor by the first term in the quotient, placing each product in its proper column, viz. x^4 under x^4, x^3 under x^3, &c. and in the same line as the corresponding term of the divisor. The work then stands as follows:

$$\begin{array}{r|l} x^2 & x^5+4x^4-2x^3+x^2-4x+6 \\ +3x & \quad +3x^4 \\ -5 & \quad\quad -5x^3 \\ \hline & x^3 \end{array}$$

Step 4. Add up the first incomplete column and divide the result by the first term of the divisor; this gives the second term of the quotient, and being placed in the bottom line completes the second column.

Step 5. The multiplication proceeds as in Step 3 and the work now stands:

$$\begin{array}{r|l} x^2 & x^5+4x^4-2x^3+x^2-4x+6 \\ +3x & \quad +3x^4+21x^3 \\ -5 & \quad\quad -5x^3-35x^2 \\ \hline & x^3+7x^2 \end{array}$$

Step 6. Proceed as in Steps 4 and 5, and the completed work will be:

$$\begin{array}{r|l} x^2 & x^5+4x^4-2x^3+x^2-4x+6 \\ +3x & \quad +3x^4+21x^3+42x^2+24x \\ -5 & \quad\quad -5x^3-35x^2-70x-40 \\ \hline & x^3+7x^2+14x+8 \quad \text{Quotient.} \end{array}$$

Step 7. We cannot divide any further; adding the two remaining columns we obtain the remainder, viz. $-50x-34$.

Horner's method is particularly useful when the divisor contains only two terms.

Divide $x^7-6x^5+4x^3-3x+5$ by $x+3$.

$$\begin{array}{r|l} x & x^7 - 6x^5 + 4x^3 - 3x + 5 \\ -3 & -3x^6+9x^5-9x^4+27x^3-93x^2+279x-828 \\ \hline & x^6-3x^5+3x^4-9x^3+31x^2-93x+276 \quad \text{Quotient.} \\ \hline & -823 \quad \text{Remainder.} \end{array}$$

The following example is important:

Divide $x^3+y^3+z^3-3xyz$ by $x+y+z$.

Long Method.

$$\begin{array}{l} x+y+z \,)\, x^3 \ldots -3xyz+y^3+z^3 \,(\, x^2-xy-xz+y^2-yz+z^2 \\ \quad x^3+x^2y+x^2z \\ \quad \overline{-x^2y-x^2z \quad -3xyz} \\ \quad -x^2y \quad -xy^2- \quad xyz \\ \quad \overline{-x^2z+xy^2-2xyz} \\ \quad -x^2z \quad - \quad xyz-xz^2 \\ \quad \overline{xy^2- \quad xyz+xz^2+y^3} \\ \quad xy^2 \quad +y^3+y^2z \\ \quad \overline{-xyz+xz^2 \quad -y^2z} \\ \quad -xyz \quad -y^2z-yz^2 \\ \quad \overline{+xz^2 \quad +yz^2+z^3} \\ \quad +xz^2 \quad +yz^2+z^3 \\ \quad \overline{} \end{array}$$

Horner's Method.

$$\begin{array}{r|l} x & x^3+0x^2y+0x^2z+0xy^2-3xyz+0xz^2+y^3+0y^2z+0yz^2+z^3 \\ -y & -x^2y \quad +xy^2+xyz \quad -y^3+y^2z-yz^2 \\ +z & -x^2z \quad +xyz+xz^2 \quad -y^2z+yz^2+z^3 \\ \hline & x^2- \quad xy- \quad xz \quad + \quad y^2- \quad yz+ \quad z^2 \quad \text{Quotient.} \end{array}$$

Examples XI.

Divide:

1. $3p^5+16p^4-33p^3+14p^2$ by p^2+7p.
2. $x^5+5x^4-9x^3-38x^2-x+6$ by x^3-7x-3.
3. $-4a^2b-3ab^2+5bc$ by $-4b$.
4. $2x^5+22x^4+127x^3-127x+30$ by $x^2+11x-6$.
5. $12-8x+2x^3-7x^2+x^4$ by $2-3x+x^2$.
6. $16x^5-4x^3-8x^2+x+1$ by $4x^2-2x-1$.

7. $1+3x-24x^3+8x^4$ by $2x^2+3x-1$.
8. $8x^5-12x^3+35x^3+16x-7$ by $2x^2+3x-1$.
9. $5x^6-6x^5+1$ by $1-2x+x^2$.
10. $5x^4-4x^3+3x^2+22x+55$ by x^2-3x+5.
11. $14x^5+33x^3+25x^2+51-68x-34x^4$ by $2x^2+3-4x$.
12. $x^4+18x^3-216x-3888$ by $x^2+12x-108$.

---

13. Prove that the sum of the cubes of two numbers divided by their sum equals the sum of their squares diminished by their product.
14. What is the quotient when x^6-y^6 is divided by
$$(x-y)(x^2+xy+y^2)?$$
15. Find the quotient and remainder when
$6x^4-5x^3y+4x^2y^2-5xy^3+6y^4$ is divided by $x^2-2xy-7y^2$.
16. Find an expression such that when it is divided by b^2-ab+a^2 the quotient is $b^2-2ab+a^2$ and the remainder $2a^3(b-a)$.
17. Divide each of the following expressions by $a-b$:
(i) a^2-b^2+a-b; (ii) $a^3-b^3+ab(a-b)$;
(iii) $x(a-b)+y(a-b)$.
18. If $2x^4+x^3-6x^2-ax-2$ is exactly divisible by x^2+2x+1, find the value of a.
19. Divide each of the following expressions by $x-y$:
(i) $a(x-y)+b(x-y)+c(x-y)$;
(ii) $a(x^2-y^2)+b(x^2-2xy+y^2)$;
(iii) $x^3+x^2+x-y^3-y^2-y$.
20. Add together $1+4x-12x^2$, $12x^2-2+6x$, $-10x-13x^2+2$, and $3x-20x^2-4$. Multiply the result by $3-2x$ and divide the product by $1-x+11x^2$.
21. Find the quotient and remainder when x^3+y^3+8 is divided by $x+y+2$.
22. Divide $x^6-4y^4x^4-y^2x^3+4y^6$ by x^2-y.
23. If $x-3$ will divide $x^5-cx+27$ without remainder, what is the value of c?
24. Divide $x^3+3xy+y^3-1$ by $x+y-1$.
25. Simplify $\dfrac{x^4-3x^3+4x^2-x+5}{x^2+2x-3}$.
26. Divide $2x^3+4x^2+8x+18$ by $x+2$.
What is the value of the dividend if $x+2=0$?
27. Divide $x^4-4x^3-12x-9$ by x^2-2x-3.

28. Divide x^6-1 by x^2-x+1.

29. Find the quotient and remainder when
$x^4-3x^3+5x^2+13x-4$ is divided by x^2-2x+1.

30. Simplify $\dfrac{(3x+4y)^2+(4x-3y)^2}{x^2+y^2}$.

31. Solve $a(x-2b)+b(x-2a)=(a-b)^2$.

32. Solve $a(x+a-b)=b(x-a+b)$.

33. Solve $(b+c)(x-a)-(c+a)(x-b)=(a+b)(x-c)$.

34. Solve $a(x-a)-b(x-b)=a-b$.

35. Solve $a(x-3b)+2b(x-a+b)=2(a^2+2b^2)$.

36. Prove that $x^m \div x^n = x^{m-n}$ when m and n are positive integers and m is greater than n.

37. Find the remainder when x^3+px^2+qx+r is divided by $x+a$. What does the dividend become when $x+a=0$? 113.

38. Divide
$1-x+y-xy-x^2y^2-x^2y^3+x^3y^2+x^3y^3$ by $1-x+xy-x^2y$.

39. Solve $\left.\begin{aligned} ax-cy &= ab-c^2 \\ cx-by &= c^2-ab \end{aligned}\right\}$.

40. Divide $x^3-ax^2+a(b+c)x-abc-bx^2-cx^2+bcx$ by $x-a$.

41. Solve $\left.\begin{aligned} x-ny &= (1-n^2)a \\ ax-by &= a^2-b^2 \end{aligned}\right\}$.

42. Show by division that $\dfrac{1-x}{1+2x} = 1-3x+6x^2-12x^3+\&c.$

43. Solve $\left.\begin{aligned} bx-ay &= 2a+b \\ &= 2x+y \end{aligned}\right\}$.

44. Divide $x^4-56x+15$ by $x^2+4x+15$ and verify your answer when $x=10$.

45. Solve $a(x-1)+b(x-a)=x(b-1)$.

46. Find what $12x^5-20x^4-13x^3+32x^2-5x-3$ must be divided by in order that the quotient may be $3x^3-5x+2$ and the remainder $4x-1$.

47. Divide
$x^6-2x^5-(a^2-1)x^4+2a^2x^3+a^2(a^2-1)x^2-2a^4x+a^4$ by x^2-2x+1.

48. Prove that $ax^5+bx^4+cx^3+cx^2+bx+a$ is exactly divisible by $x+1$; and is divisible by $x-1$ if $a+b+c=0$.

49. Divide
$3\{a^2(b+c)+b^2(c+a)+c^2(a+b)\}-2(a^3+b^3+c^3)-12abc$
by $b+c-2a$.

50. Divide $x^2y+xy^2-x^2-y^2-3xy+2x+2y-1$ by $x+y-1$.

CHAPTER XII

INVOLUTION AND EVOLUTION

83. **Involution** is the process of finding powers of quantities.

Examples. $(3x^2)^4 = 3x^2 \times 3x^2 \times 3x^2 \times 3x^2$

$= 3^4 x^{2+2+2+2}$

$= 3^4 x^{2\times 4}$

$= 81x^8.$

$(-3x^2)^3 = (-3) \times (-3) \times (-3) \times x^2 \times x^2 \times x^2$

$= +3^2 \times (-3) \times x^{2+2+2}$

$= -3^3 x^{2\times 3}$

$= -27x^6.$

Examples XII a.

Find the value of:

1. $(2x)^3$. 2. $(-2x)^5$. 3. $(-2x)^6$. 4. $(2x)^6$.
5. $(2x^2)^3$. 6. $(-2x^2)^3$. 7. $(-2x^2)^6$. 8. $(2x^2)^6$.
9. $(3a^2b)^4$. 10. $(-3ab^2)^3$. 11. $(-3p^2q^3)^5$.
12. $(2p^3)^3 \times (3p^3)^2$. 13. $(-2ab)^5$. 14. $(x^5y^7)^{10}$.
15. $(x^2y)^4 \times (xy^2)^3$. 16. $(-3ab^2c^3)^4$. 17. $(4a^2bc^3)^6$.
18. $(2a^2b)^5 \div (2ab^2)^3$. 19. 3^{20}. 20. 2^{30}.
21. What is the quantity whose square is $49x^2$?
22. What is the quantity whose cube is $-27x^6$?
23. What is the value of $(-1)^{19}$? What is the value of $(-x^7)^{24}$?
24. Find the value of $(2x^p)^3$, $(-2x^p)^3$, and $(x^p)^q$.
25. What is the value of $(-x^a)^b$?

84. The student will now have found out the following rules:

(i) **An even power of a negative quantity is positive, an odd power of a negative quantity is negative.**

(ii) $(x^p)^q = x^{pq}$ or in words: **When a power is itself raised to a power, the result is a power whose index is the product of the two original indices.**

(iii) To raise a single term to any power, determine the sign of the result by rule i, raise the coefficient to the required power by multiplication, and multiply the index of every factor in the term by the index of the required power.

85. If $x^n = y$, then x is said to be the n^{th} **root** of y. That is, the n^{th} root of a quantity is a quantity whose n^{th} power equals the first quantity. Thus since $7^2 = 49$, 7 is the second or square root of 49; since $(x^2)^3 = x^6$, x^2 is the third or cube root x^6; similarly as $(2x^2)^4 = 16x^8$, therefore $2x^2$ is the fourth root of $16x^8$.

The symbol $\surd$ is used to denote a root, e. g. $\sqrt[3]{8}$ means cube root of 8, $\sqrt[5]{243x^{10}}$ means the fifth root of $243x^{10}$. The small figure placed before the symbol is called the index or **exponent** of the root; the symbol itself is called the **radical sign.**

In the case of the square root the index is omitted; and when there is no index mentioned the word root is always taken to mean square root; thus $\sqrt{x}$ is read root x, and means square root of x.

Evolution is the process of finding any root of a quantity.

As evolution is the exact reverse of involution, the rules for evolution are the reverse of the rules for involution.

Examples XII b.

Find the values of the following roots and in each case prove the answer:

1. $\sqrt{4x^2}$.
2. $\sqrt[3]{8y^3}$.
3. $\sqrt[4]{16x^8y^4}$.
4. $\sqrt{\dfrac{a^4b^6}{x^4}}$.
5. $\sqrt[6]{\dfrac{64x^6}{y^{12}}}$.
6. $\sqrt[3]{\dfrac{81a^6b^3}{8\times 9}}$.
7. $(\sqrt{7})^2$.
8. $\sqrt{-4x^2}$.
9. $\sqrt[3]{-27p^3}$.
10. $\sqrt{\dfrac{4x^{2a}}{y^4b}}$.
11. $\sqrt[5]{-243x^{15}}$.
12. $\sqrt{81a^4b^6}$.
13. $(-2)^2 = 4$, what is $\sqrt{4}$?
14. $(-2)^3 = -8$, what is $\sqrt[3]{-8}$?

15. $(+2)^2 = 4$, what is $\sqrt{4}$?
16. $(+2)^3 = 8$, what is $\sqrt{8}$?
17. Find a quantity which when multiplied by itself equals -1.
18. Between what whole numbers does $\sqrt{741}$ lie?

86. From the last set of examples the following rules may be discovered.

(i) An even root of a positive quantity may be either positive or negative; thus $\sqrt{4}$ may be $+2$ or -2; this is usually written $\sqrt{4} = \pm 2$.

An even root of a negative quantity cannot be found, and is said to be an **impossible** or **imaginary** quantity. The only thing known about $\sqrt{-2}$ is that its square is -2.

An odd root of a positive quantity is positive, an odd root of a negative quantity is negative.

(ii) **A root of a power is a power whose index is found by dividing the index of the original power by the index of the root.**

(iii) A root of a number must be found by trial. It is shown in Arithmetic how to find the square and cube roots of any number. It will be shown in a later chapter how any root of a number may be found approximately.

(iv) To find a root of any single term, determine the sign by rule i, the coefficient as suggested in rule iii, and divide the index of every letter factor of the term by the index of the root.

Involution of the Sum or Difference of Two Terms.

87. By actual multiplication it can be shown that

$$(x+y)^2 = x^2 + 2xy + y^2$$

and

$$(x-y)^2 = x^2 - 2xy + y^2.$$

These are formulae and must be learnt by heart. It is useful also to recollect the corresponding rules in full.

The square of the sum of two quantities is equal to the sum of the squares of the quantities increased by twice their product.

The square of the difference of two quantities is equal to the sum of the squares of the quantities diminished by twice their product.

88. The two formulae of the previous section are really only one, and either of them can be regarded as a particular case of the other.

$$(x+y)^2 = x^2+2xy+y^2.$$

y may stand for any quantity; we may therefore replace it by $-y$. The formula then becomes

$$\{x+(-y)\}^2 = x^2+2x(-y)+(-y)^2;$$

i. e. $$(x-y)^2 = x^2-2xy+y^2.$$

89. There are two simple formulae for the cube of the sum or difference of two quantities.

$$(x+y)^3 = (x+y)^2 \times (x+y)$$
$$= x^3+3x^2y+3xy^2+y^3.$$

$$\begin{array}{r} x^2+2xy+y^2 \\ x+y \\ \hline x^3+2x^2y+xy^2 \\ \quad +x^2y+2xy^2+y^3 \\ \hline x^3+3x^2y+3xy^2+y^3 \end{array}$$

Similarly or by substituting $-y$ instead of y we have $(x-y)^3 = x^3-3x^2y+3xy^2-y^3$.

There is another form for the 'cube' formulae which is often more useful.

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$= x^3+y^3+3xy(x+y);$$

and similarly

$$(x-y)^3 = x^3-y^3-3xy(x-y).$$

90. Any higher power of the sum or difference of two quantities must be found by actual multiplication, which can often be shortened by using the index law for multiplication.

Thus $(a-b)^4 = (a-b)^2(a-b)^2$
$$= (a^2-2ab+b^2)^2$$
$$= a^4-4a^3b+6a^2b^2-4ab^3+b^4 \quad \text{(see § 91)},$$

$$(2a+3b)^5 = (2a+3b)^2\,(2a+3b)^2\,(2a+3b)$$
$$= (4a^2+12ab+9b^2)^2\,(2a+3b)$$
$$= (16a^4+96a^3b+216a^2b^2+216ab^3 + 81b^4)\,(2a+3b) \quad \text{(see § 91)};$$

and the work can now be finished by multiplication.

Examples XII c.

Find the value of

1. $(2x+3)^2$.
2. $(x-3y)^2$.
3. $(3a+2b)^2$.
4. $(2x+3a)^2$.
5. $(3x-2a)^2$.
6. $(3p+4q)^2$.
7. $(x^2-y^2)^2$.
8. $(3x^2+2y^2)^2$.
9. $(4abc-2xy)^2$.
10. $(3x^2+4yz)^2$.
11. $(x-2pq)^2$.
12. $(3ab+5bc)$.
13. $(a-b)^3$.
14. $(a+b)^3$.
15. $(p-2q)^3$.
16. $(2p+q)^3$.
17. $(3p-4a)^3$.
18. $(2x+3y)^3$.
19. $(ab-3)^3$.
20. $(2x-5z)^3$.
21. $(2ab-3xy)^3$.

Find by a graphical method:

22. $(a+b)^2$.
23. $(x+3)^2$.
24. $(2x+3y)^2$.
25. $(x-y)^2$.
26. $(x-4)^2$.
27. $(a+b+c)^2$.

Find, if possible, the square root of

28. $p^2+2pq+q^2$.
29. $4p^2-4pq+q^2$.
30. $4x^2+12xy+9y^2$.
31. $4x^2-12xy+9y^2$.
32. $4a^2-16ab+9b^2$.
33. $9a^2-30ab+25b^2$.
34. $9x^2+14y^2-12xy$.
35. $16x^2+40xy+25y^2$.

What must be added to the following expressions to make them exact squares?

36. a^2+2ab.
37. a^2+b^2.
38. $4x^2-4xy$.
39. x^2-6x.
40. $9x^2+4y^2$.
41. $a^2b^2+2ab^2c$.

Use the formulae for $(a+b)^2$ and $(a-b)^2$ to obtain

42. 101^2.
43. 997^2.
44. 96^2.
45. 57^2.
46. 72^2.
47. 89^2.

Find the value of

48. $(x+y)^4$.
49. $(2x-y)^5$.
50. $(x+2y)^6$.

91. The formulae of § 89 are true for any quantities; we can therefore deduce a formula for the square of the algebraic sum of any number of quantities.

We have $(x+y)^2 = (x^2+2xy+y^2)$.

Put $x = a$, $y = b+c+d$;

then $(a+b+c+d)^2 = a^2+2a(b+c+d)+(b+c+d)^2$. (i)

In exactly the same way it can be shown that

$$(b+c+d)^2 = b^2+2b(c+d)+(c+d)^2.$$

Hence substituting in i, we have the formula

$$\begin{aligned}(a+b+c+d)^2 = a^2&+2a(b+c+d)\\&+b^2+2b(c+d)\\&+c^2+2cd+d^2.\end{aligned}$$

Hence we have the rule.

The square of the sum of any number of terms equals the square of the first term + twice the first term multiplied by each of the following terms + the square of the second term + twice the second term multiplied by each of the following terms + the square of the third term + twice the third term multiplied by each of the following terms, and so on till we come to the square of the last term.

Thus $(2x+3y-4z)^2 = 4x^2+12xy-16xz$
$+9y^2-24yz+16z^2.$

Careful attention must be paid to the sign of each of the products.

Examples XII d.

Find the square of each of the following expressions:

1. $a+b-c$.
2. $x-y-z$.
3. $-x-y-z$.
4. $2a+3b+1$.
5. $2x+3y-4z$.
6. $3y-4x+2z$.
7. x^2+x-1.
8. x^2-x-2.
9. $3x^2+2x+5$.
10. $2a+3b-c+2d$.
11. x^3-2x^2+3x+1.
12. $3p+2q+r-5s$.

When $x=(a-b)$ and $y=(a+b)$, find the value of each of the following:

13. $2x^2+3y^2$.
14. $3x^2-2y^2$.
15. $4x^2-9y^2+xy$.
16. x^2-y^2.
17. $8x^2+4y^2$.
18. $(3x+2y)^2$.

Find the values of the following expressions when $x=a$, $y=(b-c)$:

19. $(3x-2y)^2$.
20. $(x+2y)^2$.
21. $(2x-3y+a-2b)^2$.
22. Simplify $(x+y)^2+(x-y)^2$ and $(x+y)^2-(x-y)^2$.
23. Prove that $(9x-1)^2-2(7x+3)^2$ equals $(x-9)^2-2(3x+7)^2$.
24. If $p=b+c-2a$ and $q=c+a-2b$, show that
$p^2+q^2-2pq=9(a-b)^2$.

Evolution of expressions containing more than one term.

92. It is shown in § 87 that $(a+b)^2 = a^2+2ab+b^2$, hence reversing the operation $\sqrt{a^2+2ab+b^2} = a+b$. Therefore if an expression consists of three terms of which two are exact squares and the third is twice the product of the square roots of the other two, then the square root of the whole expression can be found.

Examples. $16x^2+24xy+9y^2 = (4x)^2+(3y)^2+2(4x)(3y)$

$\therefore \sqrt{16x^2+24xy+9y^2} = 4x+3y.$

$4a^2+25b^2-20ab = (2a)^2+(5b)^2-2(2a)(5b)$

$\therefore \sqrt{4a^2+25b^2-20ab} = 2a-5b.$

The sign $\pm$ should be placed before each square root, but it is often omitted.

93. In a similar way by reversing the rule of § 91 it is sometimes possible to find the square root of expressions containing more than three terms.

Example. Find the square root of

$$4x^2+12xy-16xz+9y^2-24yz+16z^2.$$

We see that there are three square terms, $4x^2$, $9y^2$, $16z^2$. This suggests that the square root is $2x \pm 3y \pm 4z$. Squaring this last expression, we get the right product terms by taking the second term positive and the third term negative.

Hence the square root $= 2x+3y-4z$.

The solution can also be arranged thus:

Expression $= (2x)^2+4x(3y-4z)+(3y-4z)^2$.

$\therefore$ sq. root $= 2x+3y-4z$.

Examples XII e.

Find when possible the square roots of the following expressions:

1. $p^2-2pq+q^2$.	2. $4x^2-4x+1$.	3. $9x^2+12xy+4y^2$.
4. $9x^2+16y^2$.	5. $x^2+2xy-y^2$.	6. $4x^2-8ax+4a^2$.
7. $4a^2-12ab+9b^2$.	8. $y^2-10yz+25z^2$.	9. $49a^2+112ab+64b^2$.
10. a^2+4b^2-4ab.	11. $8ab+4b^2+4a^2$.	12. x^4-10x^2+1.
13. a^4+4+4a^2.	14. $1-2x+x^2$.	15. $z^2-2zx+x^2$.

16. $a^6-6a^3b^3+9b^6$. 17. $p^{10}-18p^5+81$. 18. b^4+b^2+1.
19. $(x+y)^2-2(x+y)(a+b)+(a+b)^2$.
20. $x^2-2xy+y^2+2(x-y)a+a^2$.
21. $a^2-2ab+b^2-2ac+c^2+2bc$.
22. $4a^2+9b^2+16c^2+12ab-16ac-24bc$.
23. $x^2+4xy-6xz+4y^2-12yz+9z^2$.
24. $a^2+b^2+c^2+2ab-2bc+2ac$.
25. $4x^4-12x^3+13x^2-6x+1$.
26. $9p^2-6pq+q^2-24p+8q+16$.
27. $x^2+y^2+z^2-2xy-2xz-2yz$.

Examples XII f.

1. If $x=\sqrt{3}$, $y=\sqrt{5}$, what is the value of x^2y^2? What is the value of xy?

2. What is the square of each of the following:
(i) $\sqrt{7}$; (ii) $\sqrt{9\frac{1}{2}}$; (iii) $\sqrt{9017}$.

3. Prove that $\sqrt{x}\times\sqrt{y}=\sqrt{xy}$.

4. Show that $(2-\sqrt{3})^2=7-4\sqrt{3}$.

5. Find the value of $(\sqrt{5}+2)(\sqrt{5}-2)$.

6. Find correct to two decimal places $(\sqrt{17})^2-(\sqrt{15})^2$.

7. What must $\sqrt{7}$ be multiplied by to give
(i) 7; (ii) $\sqrt{21}$; (iii) 14?

8. If $x=\dfrac{-1+\sqrt{13}}{2}$, show that $x^2+x-3=0$.

9. If $x=5$ and $y=3$, find the value of
$$x\sqrt{x^2-8y}+y\sqrt{x^2+8y}.$$

10. If $a=3$, $b=-2$, find the value of $\sqrt{a^4-5b^3}$.

11. If $a=1$, $b=3$, $c=-5$, $d=0$, find the value of
$$abc+\frac{c}{ab}+abd+\sqrt{c^2-b^2}.$$

12. If $x=8$, $y=6$, $z=0$, $\omega=4$, find the value of
$$\frac{xy}{4\omega}-\frac{7}{52}xyz^2-\sqrt[3]{\frac{xy^3-z}{\omega^3}}.$$

13. Show that $x=12$ is a root of the equation
$$\sqrt{2x+1}-\sqrt{x+4}=1.$$

14. What is the value of x^2y when $x=3$, $y=2$? Hence show that $\sqrt{18}=3\sqrt{2}$.

15. Show that $\sqrt{50} = 5\sqrt{2}$.

16. Which is the greater $3\sqrt{2}$ or $2\sqrt{5}$?

17. What must $\sqrt{x}-\sqrt{y}$ be multiplied by to give $x-y$ as product?

18. Find the value of $\sqrt{x}$ when x has the following values: 0, 1, 1·21, 1·44, 1·69, 1·96, 2·25. Hence draw the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 2{\cdot}25$. By means of the graph find $\sqrt{1{\cdot}8}$ and verify by calculation.

19. Draw the graph of $y = x^2$ from $y = 0$ to $y = 4$; and use it to find the value of $\sqrt{3}$ and $\sqrt{2}$.

20. Simplify $\dfrac{7-5}{\sqrt{7}+\sqrt{5}} + \dfrac{6-5}{\sqrt{6}-\sqrt{5}}$, and hence find correct to two decimal places the value of $\dfrac{2}{\sqrt{7}+\sqrt{5}} + \dfrac{1}{\sqrt{6}-\sqrt{5}}$.

Examples XII.

1. A regiment can be formed into a solid square with x men in each rank. How many men are there in the regiment? How many additional men are required to form a solid square with 5 more men in each rank?

2. A man who is x years old has a son y years younger than himself. He leaves the son as many pounds for each year of his son's age as there are years in the son's age. How much does the son receive?

3. Prove that the difference of the squares of any two consecutive odd numbers can be divided by 8.

4. A man buys x articles at x pence each. If he spends $4a^2 - 12ab + 9b^2$ pence, what is the value of x?

5. A man has x gold coins and y silver coins. Each gold coin is worth as many shillings as there are silver coins, and each silver coin as many pence as there are gold coins. What is the value of the coins in shillings?

6. The area of a field is $4a^2 + 12ab + 9b^2$ square yds. What is the length of a side?

7. Simplify $(y^2)^n (xy)^{m-n} \div (x^2)^m \left(\dfrac{y}{x}\right)^{m+n}$.

8. Simplify $\left(\dfrac{x^{14}}{x^5}\right)^4 \div \left(\dfrac{x^{13}}{x^5}\right)^5$.

9. Find the square root of $x^{12}(x^6-2)^2+4(x^6-1)^2$.

10. Extract the square root of

$$4x^6+16x^5+28x^4-39x^3-36x+36.$$

11. Extract the square root of

$$6ax(x^3-a^2b)+x^3(x^4-2a^2bx+9a^3)+a^4b^3.$$

12. If $x=3$, $y=-2$, $z=5$, find the value of

$$y\sqrt{z^3-4y^3}-5y^3(3xz-2y^3)-16x^3y^3.$$

13. Simplify $\left(\frac{a^2b}{c^3}\right)^2\times\left(\frac{bc^2}{a^3}\right)^3\div\left(\frac{b+c}{a}\right)^2$.

Find the square root of each of the following expressions: §93.

14. $4x^4-20x^3+13x^2+30x+9$.

15. $16a^4-8a^3-55a^2+14a+49$.

16. $9a^4-12a^3+22a^2-12a+9$.

17. $16x^4-24x^3y-7x^2y^2+12xy^3+4y^4$.

18. $4(x-1)(x^2-1)+9x^2$.

19. $9x^4-6x^3y-11x^2y^2+4xy^3+4y^4$.

20. To the square of double the product of any two consecutive numbers add the square of the sum of the numbers. Prove that the result is always a perfect square.

21. Simplify $(a-b+c)^2+(a+b-c)^2-4(b-c)^2$.

22. Express in algebraical symbols: The difference of the cubes of any two numbers exceeds the cube of their difference by three times their product multiplied by their difference. Prove that the statement is true.

23. Find the sum of $(\overline{a+b}+\overline{c-d})^2$, $(\overline{a+b}-\overline{c-d})^2$, $(\overline{a-b}+\overline{c+d})^2$ and $\{-(a-b)+(c+d)\}^2$.

24. Represent the following statement symbolically:

The n^{th} power of a certain number exceeds the n^{th} root of the number by the q^{th} root of its p^{th} power.

25. If $x=-p\pm\sqrt{p^2-4q}$, find correct to two places of decimals the value of x when

(i) $p=1$, $q=-2$;
(ii) $p=-1$, $q=-2$;
(iii) $p=-2$, $q=-8$;
(iv) $p=5$, $q=3$;
(v) $p=-6$, $q=7$;
(vi) $p=10$, $q=-10$.

26. If a line of length x inches is cut from a line whose length is a inches, what is the length of the remainder?

What is the answer when $a=1$, $x=\frac{\sqrt{5}-1}{2}$?

In this case prove that $a(a-x)=x^2$.

CHAPTER XIII

REVISION CHAPTER

94. Definitions. The words term (§ 4) and factor (§ 5) must be carefully distinguished. Note that a term may contain several factors and that a factor may contain several terms, e.g.

$$2ab+3ax-4bc$$

where each term consists of three factors and $(3a+b)(a+3b)$ where each factor consists of two terms.

An expression consisting of one term is called a **monomial** expression, of two terms a **binomial** expression, one of three terms is a **trinomial**, and of more than three a **multinomial** expression.

The definitions of 'power' and 'index' (§ 12) must be known thoroughly. The two are often confused, but it ought to be easy to recollect that the 'power' is the result of the multiplication, and that the 'index' merely shows how many factors are multiplied together. Thus x^3 is the third power of x and equals $1\times x\times x\times x$; the index is 3 and shows that there are three factors equal to x. Each letter factor of a term is called a **dimension**, and the number of dimensions is the **degree** of a term; thus $5a^3b$ which means $5aaab$ is of four dimensions, i.e. is of the fourth degree.

The **degree** of an expression or equation is determined by the degree of the term of highest dimensions in the expression or equation. Thus $2x^3-3x^2+4x+7$ is of the third degree.

A **homogeneous** expression is one in which all the terms are of the same dimensions; the expression

$$3x^4-4x^3y+6x^2y^2-3xy^3+y^4$$

is homogeneous since each term is of the fourth degree.

Equations and expressions of the first degree are often said to be **linear**; if of the second degree they are said to be **quadratic**, if of the third degree **cubic**.

95. Signs of equality and inequality:

$=$ means 'equals'.

$>$ means 'is greater than'.

$<$ means 'is less than'.

If these signs are crossed out they have the negative meaning, thus

$\neq$ means 'is not equal to'.

$\not>$ means 'is not greater than'.

$\not<$ means 'is not less than'.

$\equiv$ is often used to signify 'identically equals' and shows that the equation is an identity.

Note that both $>$ and $<$ point towards the smaller quantity.

96. Positive and Negative Numbers. The whole of Chapter III is important. Note the three ways of regarding negative numbers:

(i) as the result of subtracting positive numbers from nothing;

(ii) as obtained by counting from nothing in a direction opposite to that in which positive numbers are counted;

(iii) as having a meaning exactly opposite to the meaning attached to positive numbers.

These three points of view are quite consistent and all lead to the result that

$$+a+(-a)=0.$$

This may be regarded as the essential property of negative numbers.

The fact that x is positive is often denoted by $x>0$; similarly $x<0$ is another way of saying that x is negative.

97. If a and b are two numbers such that $a-b$ is positive, then a is said to be greater than b, i. e. if $a-b>0$ then $a>b$.

It follows that any positive number is greater than any negative number, e. g. $\cdot 00001 > -1000000$.

Fundamental Laws.*

98. Algebra may be divided into two divisions, Arithmetical Algebra and Symbolical Algebra. In Arithmetical Algebra letters are used, but the letters are supposed to have such values as make

* Sections 98–101 may be omitted at the first reading.

the operations arithmetically intelligible. This is the kind of Algebra we have used in Chapters I and II. In Arithmetical Algebra the letters are supposed to stand for positive numbers, also in such an expression as $x-y$ it is understood that x stands for a greater number than y.

By our knowledge of Arithmetic it is fairly easy to prove the simple laws of Arithmetical Algebra, such as $a+b=b+a$ and $ab=ba$. In Symbolical Algebra, these laws are assumed to be true for all values of the letters; by these assumptions extended meanings are found for the operations of addition, multiplication, subtraction, and division. When once these extended meanings have been grasped, there is no need to keep up the distinction between the two kinds of Algebra.

In Chapter IV we considered special cases and so found extended meanings for the simple operations, but it was pointed out that general rules cannot be proved by taking a few numerical examples. We will now show how these same extended meanings can be found by assuming that the laws of Arithmetical Algebra are to be true whatever values the letters may have.

99. Laws of Association, e.g. $a+(b-c)=a+b-c$ and $a-(b-c)=a-b+c$.

Arithmetical Algebra.

a, b, c must all be positive and b must be greater than c.

$a+(b-c)$ means that to a we are to add the excess of b over c. If we have $a+b$ we have added c too much, therefore c must be subtracted.

$$\therefore \quad a+(b-c)=a+b-c. \qquad \text{(i)}$$

Again $a-(b-c)$ means that from a we are to subtract the excess of b over c. If we have $a-b$ we have subtracted c too much, therefore c must be added.

$$\therefore \quad a-(b-c)=a-b+c. \qquad \text{(ii)}$$

In exactly the same way we can prove that

$$a+(b+c)=a+b+c \qquad \text{(iii)}$$

and

$$a-(b+c)=a-b-c. \qquad \text{(iv)}$$

These four formulae together form the Law of Association for Addition and Subtraction.

(ii) Symbolical Algebra.

Assume the Law of Association to be true for all values of the letters, and accept as true the results this leads to.

e.g. assume $a+(b-c)=a+b-c$.

Put $b = 0$, then $a+(-c) = a-c$. (i)

Again, assume $a-(b-c) = a-b+c$.

Put $b = 0$, then $a-(-c) = a+c$. (ii)

Hence, if the Law of Association is always to be true, the result (i) shows that the addition of a negative quantity must be equivalent to the subtraction of the corresponding positive quantity; while result (ii) shows that the subtraction of a negative quantity must be equivalent to the addition of the corresponding positive quantity. These meanings which must, therefore, be attached to the operations of adding and subtracting negative quantities, agree with what we had already found out by experiment in Chapter IV.

They can also be deduced by assuming the **Law of Commutation**, viz. $a+b = b+a$.

100. Law of commutation for addition and subtraction, i.e. $a+b+c+d = a+c+d+b = +c+d+b+a$, &c.

It is sufficient to discuss the case of two quantities, viz.

$$a+b = b+a$$

Arithmetical Algebra. If a and b are positive integers (i.e. whole numbers), $a+b$ implies that there are two groups of objects, one containing a objects, the other b objects, and that we wish to find what number we should reach if we count the number of

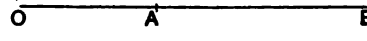

FIG. 25.

objects in the two groups together. It is clear that the number is the same whether we start counting the a group or the b group. Hence, when a and b are positive integers, $a+b = b+a$.

Graphical method. Draw OA a units long and produce to B so that AB is b units long.

If we measure from O the length of OB is $a+b$ units, but if we measure from B the length is $b+a$ units.

But the length of the line is the same whichever way we measure.

$\therefore$ $a+b = b+a$.

This method applies even if a and b are not integers, provided they are both positive.

We can now prove the law for more than two quantities.

$$a+b+c+d = a+c+d+b \text{ changing two letters}$$
$$= c+a+d+b \quad ,, \quad ,,$$
$$= c+d+a+b \quad ,, \quad ,,$$
$$= c+d+b+a \quad ,, \quad ,,$$

Symbolic Algebra. Assume $a+b=b+a$ as true for all values of a and b.

Suppose $b=-x$ where $x>0$ but $<a$;
then $a+(-x)=-x+a$,
but $a = x+(a-x)$ by the Association Law.
$\therefore \quad a+(-x) = -x+\{x+(a-x)\}$
$= -x+x+(a-x)$ by the Association Law
$= a-x.$ (i)

Suppose $b=-x$ where $x>a$ and $a>0$;
then $a = x-(x-a)$ by the Association Law.
$\therefore \quad a+(-x) = -x+a$
$= -x+\{x-(x-a)\}$
$= -x+x-(x-a)$ by the Association Law
$= -(x-a).$ (ii)

Results (i) and (ii) give us the meaning to be attached to the addition of a negative quantity, and agree with the meaning found from the Association Law in § 99 and with the meaning found by experiment in § 22.

Since $a-x=a+(-x)$, and since the Commutation Law has been assumed to be true for all values, it follows that it is also true for combined addition and subtraction, e.g.

$$a+(-b)+(-c)+d = (-c)+d+(-b)+a.$$
$$\therefore \quad a-b-c+d = -c+d-b+a,$$

assumed by (i) above.

101. Law of Commutation for multiplication, i.e. $ab=ba$.

Arithmetical Algebra. This law has been proved, when a and b are positive, by the graphical method in § 11, Ex. 5.

Symbolical Algebra. Assume $ab=ba$ in all cases.

Suppose ab means a multiplied by b,
and that $a=-x$, where x is positive;
then $(-x)b$ means $-x$ multiplied by b
$= (-x)+(-x)+(-x) \ldots\ b$ terms
$= -x-x-x \ldots\ b$ terms
$= -bx;$ (i)

but since $ab=ba$ in all cases,
$\therefore \quad (-x)b = b(-x)$,

i. e. b multiplied by $(-x) = (-x)\,b$
$= -bx$ by (i).

Hence we have the meaning to be attached to multiplication by a negative quantity, viz. multiply by the corresponding positive quantity but change the sign of the product.

This, again, agrees with the rule already found in Chapter IV, § 29.

In a similar way when other laws * of Arithmetical Algebra are assumed to be true in all cases, they can be shown to lead to the same interpretation of the meaning of addition, multiplication, &c., of negative quantities as have been found in Chapter IV.

102. Brackets. Be careful to notice that brackets are not necessarily to be removed whenever they occur. When brackets are used to enclose the coefficient of a power of x they are usually to be kept; when any power of x comes inside them it is usually more convenient to remove the brackets.

Example. *Divide* $x^3-(a+b+c)\,x^2+(bc+ca+ab)\,x-abc$ *by* $x-a$.

Keep the brackets.

$$x-a \;\big|\; x^3-(a+b+c)\,x^2+(bc+ca+ab)\,x-abc \;\big|\; x^2-(b+c)\,x+bc$$

$$\begin{array}{l} x^3-a\quad x^2 \\ \hline -(b+c)\,x^2+(bc+ca+ab)\,x \\ -(b+c)\,x^2+(\quad +ac\quad ab)\,x \\ \hline \qquad\qquad +bc\quad x\quad -abc \\ \qquad\qquad +bc\quad x\quad -abc \\ \hline \end{array}$$

Quotient $x^2-(b+c)\,x+bc$.

Examples XIII a.

Revision Examples on Chapters I—V

1. When $x = 2\,y = -2$, prove that

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)-\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)=\frac{(x+y)^2+(x-y)^2}{xy}.$$

* The rule in § 88 is called the Law of Distribution for Multiplication and is true for any number of factors.

2. In the expression

$$(2a+3)x^3-(4a^3-2a)x^2+(a^2-2a+1)x-a^2,$$

find the coefficients of the different powers of a; and determine the numerical value of the whole expression when $a=3$, $x=-5$.

3. If $a=1$, $b=\frac{1}{2}$, $c=0$, $d=-\frac{1}{3}$, find the numerical value of:

(i) $4ab-3bc-2bd-8ad$;

(ii) $a^3(b+c+d)+a(b^2+c^2+d^2)+\frac{1}{a}(b^3+c^3+d^3)$.

4. Simplify

$$3[ax-2\{-(a^4x^4-8)+3x+7x^2+6x^4\}]-6a^2x^2(1-ax);$$

and arrange the answer in ascending power of x.

5. Simplify

$$-5\{a-3[a-(b-c)]\}+30\{b-(c+a)\}.$$

6. Find the numerical value of

$$x^2(x+1)+x\{3y^2-y(3x-2)\}-y\{2x+y^2-(x-y)\},$$

where $x=4\frac{1}{2}$, $y=5\frac{1}{2}$.

7. Find the sum of

$$-(c-a)x+(a-b)y+(b-c)z,\quad -(a-b)x-(b-c)y+(c-a)z,$$

and $\quad (b-c)x+(c-a)y-(a-b)z$;

and calculate what it becomes when

$$a=b+1\tfrac{1}{2}=c+2\tfrac{1}{2}$$

8. Simplify the expressions:

(i) $-7a+9b-\{-(5a+4c)+4a-(4b+c)\}$

(ii) $-(4a+5b)-\{(5b-3a)+8c-(7b-5a)\}$;

and find the difference between them.

9. If $a=0$, $b=2$, $c=4$, $d=8$, find the value of

$$4\sqrt[3]{bc}-3\sqrt[3]{b^2d}+2\sqrt{ab}+\sqrt{bcd}.$$

10. Simplify $(x-y)^2+(x+y)^2-2(x-y)(x+y)$.

11. Find the value of

$$\frac{x^2}{(x-y)(x-z)}+\frac{yz}{(y-z)(y-x)}+\frac{z^2}{(z-x)(z-y)},$$

when $x=2$, $y=1$, $z=-3$.

12. Divide $a^2(b-c)+b^2(a-c)+c^2(b-a)$ by $b-c$.

13. Divide $a^3(b+c)+b^3(c+a)+c^3(a+b)+3abc$ by $a+b+c$.

14. Simplify $2a-[3b-\{5c-(3a-5b)-(2c-5a)\}+2b]+3c$.

15. Simplify

$$[(x+y)(x-2y)-(x-2y)(x-3y)+(x+3y)(x-y)]\div(x+y).$$

16. If a, b, c be any positive integers, prove that

$$(a+b)c=ac+bc.$$

17. Find the sum of

$3x^3+(5x^2y-xy^2)$, $3xy^2-(6x^2y-4y^3)$, $2x^2y-(xy^2+y^3)$,

and subtract $3x^2y-2xy^2$ from the result.

18. Express by means of algebraic symbols the following directions:

'Subtract the product of $2a+3b$ and $3a+4b$ from the product of $3a+2b$ and $4a+3b$, and divide the difference by three times the sum of a and b.'

Find the result.

19. Prove as clearly and as simply as you can that

$$5-(-2)=5+2.$$

Apply the same method to prove that $a-(-b)=a+b$.

20. Prove that $a^2+b^2 \not< 2ab$ whatever a and b may be. (Use the fact that all squares are positive.)

Verify in the following cases:

(i) $a=3$, $b=2$; (ii) $a=3$, $b=3$; (iii) $a=3$, $b=-2$.

21. Which of the following expressions are homogeneous, and which is of the highest degree?

(i) $3x^3-x^2+4x-2$;

(ii) $2a^2b-3ab^2+a^3$;

(iii) $4ab-6a^2b-2a^4-b^3$;

(iv) $3a^2b^3-2ab^2+3b^3$;

(v) $2a^3-3b^3+2c^3-4abc$;

(vi) $3x^2-2y^2+4xy$.

22. Divide $abx^2+(ac-bd)x^2-(af+cd)x+df$ by $ax-d$.

23. Divide $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc$ by $x+a$.

24. Prove that $a^3\times a^6=a^9$. What is the formula for dividing a power by another power of the same quantity?

Find the simplest forms for:

(i) $x^7\div x^3$; (ii) $x^3\div x^7$; (iii) $x^7\div y^3$; (iv) $3^7\div 3^3$; (v) $6^3\div 3^3$.

25. Explain why $a-(b-c)=a-b+c$.

Hence show that $a-(-c)=a+c$.

26. What do you understand by the terms positive and negative as applied to numbers and algebraical quantities? State and justify the rule for multiplying a positive by a negative quantity.

27. Simplify $4a-\{3a-(2a+1)\}$.

Give reasons to justify each step in the process of removing the brackets.

28. Define a factor, an index, and a homogeneous expression, and give an example of each.

103. The functional notation. As stated in § 48 any expression whose value depends on the value of x is called

a function of x. Just as we use the letters x, y, &c., when we are talking about numbers in general or about some number whose value is unknown, so we use the symbols $f(x)$, $F(x)$, $\phi(x)$ to denote functions of x. Thus, whenever any form of the letter f comes before a bracket, it is generally not to be regarded as a coefficient, but simply as a short way of writing 'function of'.

This functional notation is often used to save the repeated writing of some definite function. Thus, supposing the function $3x^2-5x+5$ were going to occur frequently, then it would save time and space to write $f(x)$; we might, therefore, write: 'Let $f(x)$ represent $3x^2-5x+5$' or 'let $f(x) \equiv 3x^2-5x+5$'. Throughout that piece of work or example $f(x)$ would then have this definite meaning.

Again, if $f(x) \equiv 3x^2-5x+5$,

then $f(a) \equiv 3a^2-5a+5$;

for if x is replaced by a on one side of the identity, the same change must be made on the other side.

Example. *If* $f(x) \equiv 4x^2-3x+2$, *find the values of* $f(3)$, $f(2)$, $f(1)$, $f(0)$, $f(-1)$.

By question $f(x) = 4x^2-3x+2$.

$$\therefore \quad f(3) = 4.3^2-3.3+2 = 29,$$
$$f(2) = 4.2^2-3.2+2 = 12,$$
$$f(1) = 4.1^2-3.1+2 = 3,$$
$$f(0) = 4.0^2-3.0+2 = 2,$$
$$f(-1) = 4.(-1)^2-3.(-1)+2 = 9.$$

104. Graphs. All the sections on graphs should be revised carefully.

Note the three kinds of graphs.

1. Graphs of statistics or of observations taken at long intervals. These graphs should be a succession of straight lines. Intermediate values cannot be found with this kind of graph.

2. Graphs of observations taken at small intervals. If it is found that a small change in x produces a small change in y we are justified in drawing a curve through the various plotted points. If the observations are correct and are correctly plotted,

it is almost always found that the graph is a neat-looking curve with no awkward bends in it. With such a curve intermediate values can be found with considerable accuracy.

3. Graphs of algebraical functions or equations. These are very much the same as the second kind, but are more regular in shape; and the accuracy of values intermediate to the plotted values depends entirely on the skill of the person drawing the graph.

The process of finding intermediate values is called **interpolation.**

105. The connexion between an equation and a graph must be kept clearly in view.

(i) If simultaneous values of x and y satisfy an equation, the corresponding points lie on a straight line or curve which is called the **graph of the equation.**

(ii) If a point lies on a straight line or curve, its co-ordinates satisfy an equation which is called the **equation of that straight line or curve.**

(iii) If a pair of values of x and y satisfy two distinct equations, the corresponding point will be the intersection of the graphs of the two equations.

(iv) If a point is the intersection of two graphs, its co-ordinates will satisfy the equations of both those graphs.

Note that one equation is sufficient to draw a graph, but that two equations are necessary to fix the position of a point.

106. Formulae. One of the most useful applications of Algebra is seen in the use of formulae. We give here a few which occur in elementary work.

Arithmetic.

P = principal, r = rate per cent, t = time in years, I = interest, A = amount.

Simple Interest $\quad I = \frac{Prt}{100}. \qquad A = P\left(1+\frac{rt}{100}\right).$

Compound Interest $\quad A = P\left(1+\frac{r}{100}\right)^t.$

Algebra.

$$(a+b)^2 = a^2+2ab+b^2.$$
$$(a-b)^2 = a^2-2ab+b^2.$$
$$a^2-b^2 = (a+b)(a-b).$$

Mensuration.

Area of rectangle = base × height, i.e. $A = bh$.

Area of triangle = $\frac{1}{2}$ base × height, i.e. $A = \frac{bh}{2}$.

Circumference of circle = 2π* × radius, i.e. $C = 2\pi r$.

Area of circle = π × square of radius, i.e. $A = \pi r^2$.

Surface of cylinder = circumference × height, i.e. $A = 2\pi rh$.

Surface of cone = $\frac{1}{2}$ circumference of base × slant height, i.e.

$$A = \pi r\sqrt{r^2+h^2}.$$

Surface of sphere = $4\pi r^2$.

Volume of cylinder = area of base × height, i.e. $V = \pi r^2 h$.

Volume of cone = $\frac{1}{3}$ area of base × height, i.e. $V = \frac{1}{3}\pi r^2 h$.

Volume of sphere = $\frac{4}{3}\pi r^3$.

Volume of prism = area of base × height, i.e. $V = Ah$.

Volume of pyramid = $\frac{1}{3}$ area of base × height, i.e. $V = \frac{1}{3}Ah$.

If a, b, c are sides of a triangle and $2s = a+b+c$, then area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.

Examples XIII b.

Find the value of $f(0), f(1), f(-1), f(3), f(-5)$ in the following cases:

1. When $f(x) = 3x^3 - 4x - 5$.
2. When $f(x) = 2x^3 - 9x + 2$.
3. When $f(x) = x(x-1)(x-2)$.
4. When $f(x) = 3(x+1)(x-3)(x+5)$.
5. When $f(x) = 2 - 3x + 4x^2 - x^3$.
6. Find the value of $f(a-x) + f(a+x)$ when $f(x) = x^2 + x + 1$.
7. Find the value of $\phi(a+x) - \phi(a-x)$ when $\phi(x) = a^2 + ax + x^2$.
8. Find a function of x of the first degree which equals 5 when $x = 1$, and equals 7 when $x = -1$.
9. If $\phi(x) = ax^2 + 2bx + c$, find the value of
$$\phi(x+1) + \phi(x-1) - 2\phi(x).$$
10. If $F(x) = x + 5$, find the value of
$$F(x+2) \times F(x-2) - \{F(x)\}^2.$$

* π is defined as being the number of times the diameter of a circle is contained in the circumference. It is not an exact number. For practical purposes π may be taken as $3\frac{1}{7}$, or more accurately 3·1416.

11. What can you say about $f(x)$ if $f(x) = f(-x)$?

12. Draw the graph of $y = f(x)$ if $f(x) = 3 - 2x^2$.

13. Draw the graph of $y = x^3$ from $x = 0$ to $x = 3$, and from the graph find the values of $(\cdot 75)^3$, $(2\cdot 6)^3$, $\sqrt{1\cdot 9}$.

14. A man starts walking at noon; he walks at the constant rate of 4 miles per hour and rests 5 minutes at the end of each fourth mile. Draw a graph by means of which the distance he has walked at any time before 4 p.m. can be found. How far has he walked at a quarter-past two?

15. The area of a triangle is 6 sq. ft.; draw a graph to show how the height varies as the base changes from 1 to 12 feet.

16. Taking π as $3\frac{1}{7}$, draw a graph by means of which the circumference of any circle may be found if the radius is not greater than 5 inches.

17. Find the volume of a sphere whose radius is 3·6 centimetres, taking π as 3·14. {The answer should be given correct to two significant figures.}

18. Find the area of the base of a cone 6 feet high if its volume equals the volume of a sphere 4 feet in diameter.

19. A cylindrical jug 8 inches high holds 1 quart of milk; find the radius of its cross-section. {1 c. ft. of water weighs 1000 oz., 1 gallon weighs 10 lb.}

20. Find correct to one decimal place the area of a triangle whose sides are 13, 15, 16 centimetres respectively.

21. A circular lawn of radius 10 yds. is surrounded by a path 6 feet wide; find the area of the path.

22. Find the simple interest on £105 at $4\frac{1}{2}$ % for 3 years.

23. Find the number of years £210 must be invested at simple interest before it amounts to £252; the rate of interest being 4 per cent.

24. Find the area of a triangle whose sides are 13, 12, 5 inches.

25. Find the area of a triangle whose sides are 5 ft. 5 in., 5 ft. 3 in., and 1 ft. 4 in.

26. If $f(x) \equiv ax^2 + bx + c$, $f(1) = -5$, $f(2) = -2$, and $f(-2) = 10$, find the values of a, b, and c.

27. If $\phi(x) \equiv ax^2 + bx + c$, $\phi(2) = 12$, $\phi(3) = 26$, and $\phi(-1) = 6$, find $\phi(x)$.

28. If $f(x)$ contains only odd powers of x, show that

$$f(x) + f(-x) = 0.$$

29. Draw the graphs of

$$y = 3x - 1,\ 2x - 3y = -4, \text{ and } 3x + 4y = 11.$$

Hence show that there are values of x and y which satisfy all three equations.

30. Draw the graph of $y = x^3 + 6x^2 + 3x - 10$. At what points does it cut the axis of x? Hence solve the equation

$$x^3 + 6x^2 + 3x - 10 = 0.$$

31. Plot the points given by the following table, taking 1 inch as the unit, and drawing a small circle round each point plotted.

x	$-\cdot5$	0	$\cdot5$	1	1·5
y	$-\cdot15$	·25	·8	1·35	1·75

32. Make a table giving the values of $x^3 - 3x$ when x has the values -1, 0, 1, 2, 3, 4. Then draw a curve that will show how $x^3 - 3x$ varies while x varies from -1 to 4. Find from your figure the value of $x^2 - 3x$ when $x = 2\frac{1}{2}$, and verify your result by substituting this value of x in the given expression. Find also for what value of x the expression is equal to 2.

Equations.

107. The distinction (§ 5) between an ordinary equation and an identity is important. As a rule ordinary equations have to be solved, that is, the values of x for which they are true have to be found. To verify that the solutions are correct the values found must be substituted in succession on both sides of the original equation (§ 57).

In practice the coefficients in an equation are often decimals. Unless the decimals are equal to very simple fractions they should not be replaced by vulgar fractions; they should be kept as they are, or may be removed by multiplying throughout the equation by some power of 10.

Example I. *Solve* $\cdot7x - \cdot6x + \cdot875 - \cdot3x = \cdot6$

Transpose $\cdot7x - \cdot6x - \cdot3x = \cdot6 - \cdot875$

Collect $-\cdot2x = -\cdot275$

Divide by $-\cdot2$ $x = \dfrac{-\cdot275}{-\cdot2}$

$= 1\cdot375$ *Answer.*

Example II. *Solve* $3{\cdot}23 - {\cdot}7({\cdot}35 - {\cdot}3x) = {\cdot}24(x - {\cdot}3)$

Remove brackets $3{\cdot}23 - {\cdot}245 + {\cdot}21x = {\cdot}24x - {\cdot}072$

Transpose $\quad {\cdot}21x - {\cdot}24x = -3{\cdot}23 + {\cdot}245 - {\cdot}072$

$\quad = -3{\cdot}302 + {\cdot}245$

Collect $\quad -{\cdot}03x = -3{\cdot}057$

Divide by $-{\cdot}03$ $\quad x = \dfrac{3{\cdot}057}{{\cdot}03}$

$\quad = 101{\cdot}9$ *Answer.*

108. We know that $4 \times 0 = 0$ and $5 \times 0 = 0$; therefore

$$4 \times 0 = 5 \times 0.$$

If we now try to divide both sides by 0, we obtain $4 = 5$ which is untrue. Hence we see that **if both sides of an equation are divided by nothing, the result obtained is not necessarily true.**

Now if a common factor contains the unknown quantity we do not know whether the value of that factor is zero or not; we must therefore never divide by a factor containing the unknown quantity unless we are sure that the factor is not zero.

Example I.

Solve $3x(x-2) - 2(x-2) = 3(x-1)(x-2)$.

Notice that $x-2$ is a factor throughout.

Wrong Method.

Divide by $x-2$ $\quad 3x - 2 = 3(x-1)$.

Remove brackets $\quad 3x - 2 = 3x - 3$.

Transpose $\quad 0 = -1$.

And we have not found the value of x and have got an impossible result.

Right Method.

Either $x - 2 = 0$, in which case $x = 2$.

Or $\quad x - 2 \neq 0$, and we may divide by it, and the same work as above leads to $0 = -1$.

So we have this result

either $x = 2$ or $0 = -1$.

The first alternative gives us the required value of x.

Example II.

$$3x(x-2)-2(x-2)=2(x-1)(x-2).$$

Either $x-2=0$, in which case $x=2$.

Or $x-2\neq 0$, and we may divide by it.

Divide by $x-2$ $\quad 3x-2=2(x-1)$.

Remove brackets $\quad 3x-2=2x-2$.

Transpose $\quad x=0$.

Hence we have

either $x=2$ or $x=0$.

Both alternatives are possible, so we have two values of x which satisfy the equation.

Verification.

If $x=2$, $\quad$ L. H. S. $=6(2-2)-2(x-2)=0$,

and R. H. S. $=2(2-1)(2-2) \quad =0$.

If $x=0$, $\quad$ L. H. S. $=0(0-2)-2(0-2)=4$,

and R. H. S. $=2(0-1)(0-2) \quad =4$.

Hence both solutions are correct.

Problems.

109. In all problems the following points should be noticed:

(i) Read the question very carefully and make quite sure what is given in the question and what is required to be found.

(ii) Recollect that x stands for a number, and that if a concrete quantity is to be found the units must be clearly stated, e. g.

Let $£x$ be the sum required.

Let x be the number of cows.

Let x years be the man's age, &c.

(iii) Do not give up because the equation cannot be found at once; very often half a dozen steps or more are necessary before the equation can be found.

(iv) If the equation is not obvious, write down in symbolic notation the various quantities that are given in the question; it will soon be found that two different symbolic expressions can be found for the same quantity; hence an equation can be written down.

(v) Recollect that as many equations must be found as there are unknown quantities.

(vi) If there are more unknown quantities than there are equa-

tions, it does not follow that the problem cannot be solved; on the contrary it can be solved in a large number of ways, see § 60.

110. Typical Problem questions. We do not intend to work these fully, but give a few hints which should enable the student to solve them without difficulty.

Number and Digit Problems. *A number of two digits is increased by* 45 *when its digits are reversed; what is the number?*

Recollect that the number 68 (for example) means $6 \times 10 + 8$ and that $365 = 3 \times 10^2 + 6 \times 10 + 5$.

Similarly, if a, b, c are the successive digits of a number, the number is $100\,a + 10\,b + c$.

The digits are called the units digit, the tens digit, the hundreds digit, &c.

In this particular question there are two unknowns, viz. the two digits, but we are told only one fact and can obtain only one equation. By the nature of the question, however, the unknown quantities are positive, integral, and less than ten. Hence all the solutions can be found.

Clock sums. *At what time between* 3 *and* 4 *do the two hands of a watch coincide?*

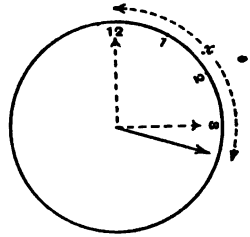

FIG. 26.

Recollect that the minute hand moves twelve times as fast as the hour hand.

Suppose the hands coincide at x minutes past 3.

Then since 3 o'clock the minute hand has moved through x spaces.

But the hour hand has moved through only $x - 15$ spaces.

$\therefore \quad x = 12\,(x - 15)$.

Fraction Problems. *Find a fraction which equals $\frac{1}{3}$ when 1 is subtracted from the numerator, and equals $\frac{1}{2}$ when 1 is added to both numerator and denominator.*

Here there are two unknowns, viz. the numerator and denominator. We are told two facts, therefore two equations can be found.

Coin Problems. *A man has 20 coins in his pocket, some of which are shillings and some are pence; their value is 6s. 3d. How many shillings has he?*

Distinguish carefully between the number of coins and the value of the coins. If two unknowns are used, two equations can be found; one depending on the number of coins, one on their value. In the value equation all the values must be expressed in the same denomination, e.g. if there are x shillings the number of them is x, but their value is $12x$ pence.

Examples XIII c.

Solve:

1. $\frac{5-3x}{4}+\frac{5x}{3}=\frac{3}{2}-\frac{3-5x}{3}$.
2. $\frac{a(a-x)}{b}-\frac{b(b+x)}{a}=x$.
3. $\frac{a(a-x)}{b}-\frac{b(a-x)}{a}=x-a$.
4. $abx=a^2(a-x)-b^2(b+x)$.
5. $\frac{1}{2}(x-\frac{1}{3})-\frac{1}{3}(x-\frac{1}{4})+\frac{1}{4}(x-\frac{1}{5})=\frac{1}{5}$.
6. $\frac{2}{5}x+\frac{15}{2}\left(1-\frac{x+1}{3}\right)+\frac{19}{10}=\frac{9}{5}\left(\frac{x+1}{6}-\frac{1}{3}\right)$.
7. $4(x+a)(x-b)+7ab=(2x+a)(2x-b)+(a+b)^2$.
8. $\frac{1}{7}(x+a+2)-\frac{1}{8}(x+2a-6)=3\frac{1}{8}(x+3a-4)$.
9. $3{\cdot}4x-{\cdot}17(x-2)=51\left(\frac{x}{5}-3\right)$.
10. ${\cdot}3(x-{\cdot}2)+3({\cdot}7x+5)=3{\cdot}6\left(\frac{x}{2{\cdot}4}-\frac{1}{2}\right)$.
11. What values of x will make the following statements true?
    (i) $4(x-3)=5(x-3)$;
    (ii) $6(x-6)=6$;

(iii) $x(x-1) = 10(x-1)$;
(iv) $(x-2)(x-7) = (2x-3)(x-7)$;
(v) $(x-5)(x-2) = (x-5)(x-4)$.

12. What values of x will make the following statements true?
(i) $3(x-5) = 4(x-5)$;
(ii) $7(x-2) = x(x-2)$;
(iii) $a(x-b) = a$;
(iv) $(x-1)(3x-4) = (2x-1)(x-1)$;
(v) $(2x-3)(x-2) = (x+12)(2x-3)$.

Solve (and in each case verify the solutions):

13. $7x+9y = x-y = 4$.

14. $\frac{7x}{3} - \frac{5y}{4} = 2\frac{1}{3}$, $4x = 3y$.

15. $ax+by = 2(a^2+b^2)$,
$\frac{x}{a} - \frac{y}{b} = \frac{a}{b} + \frac{b}{a}$.

16. $x+2y = 3a-b$,
$2x+y = 3a+b$.

17. $\frac{4}{x} + \frac{1}{y} = 1$, $\frac{8}{x} - \frac{7}{y} = -\frac{1}{6}$.

18. $\frac{a}{x} + \frac{b}{y} = c+1$, $\frac{ay-bx}{xy} = d-1$.

19. $3x+4y = 11xy$, $5x-2y = xy$.

20. $2x+\frac{1}{2}(6x-2y-1) = \frac{1}{4}+\frac{3}{4}(2y-1)$,
$\frac{1}{3}(4x+3y) = \frac{7}{10}y+1$.

21. $\cdot 12x + \cdot 23y = \cdot 857$,
$1\cdot 4x - 3\cdot 1y = -7\cdot 93$.

22. $\frac{\cdot 2}{x} = 1\cdot 3 - \frac{\cdot 45}{x}$.

23. $2x + \cdot 4y = 1\cdot 2$,
$3\cdot 4x - \cdot 02y = \cdot 01$.

24. $\frac{1}{3x} + \frac{1}{5y} = \frac{2}{9}$.
$\frac{1}{5x} + \frac{1}{3y} = \frac{1}{4}$.

25. $3x-7y+4z = 1$,
$-5x+9y-z = 22$,
$x-2y+z = 0$.

26. $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12,$

$\frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8,$

$\frac{x}{2}+\frac{z}{3}=10.$

27. $xy=3(x+y),\ xz=8(x+z),\ 7yz=9(y+z).$

28. $\frac{1}{3}x-\frac{1}{2}y+\frac{1}{4}z=7,$
$\frac{1}{2}x+\frac{1}{3}y-\frac{1}{5}z=11,$
$\frac{1}{3}x-\frac{1}{4}y+\frac{1}{5}z=5.$

29. $x+8z=2y=z+20=20y-x.$

30. $\frac{1}{2}(x-\frac{1}{2})-\frac{1}{3}(y-\frac{1}{3})=\frac{1}{2}(y-\frac{1}{3})-\frac{1}{3}(x-\frac{1}{2})=\frac{1}{6}.$

31. What do you mean by the term graph? Draw figures giving on any convenient scale the graphs of

(i) $y=-1$; (ii) $y=\frac{x}{2}-2$ from $x=0$ to $x=4$.

Hence obtain the solution of these equations if treated as simultaneous.

Solve graphically and algebraically:

32. $3y=x+2,\ y=3-2x.$

33. $2x+3y-19=0,\ 3x-2y-9=0.$

34. $3x+4y=5,\ 3y-4x=10.$

35. $2y=3(x-4),\ 3y=1-5x.$

36. $5x+3y=7,\ 7y-8x=6.$

37. $2x+y=0,\ \frac{1}{2}y-3x=8.$

38. $y-1\frac{1}{2}=3x-2,\ 2x-1\frac{1}{2}=3-y.$

39. $7x-9y=12,\ 4x+5y=10.$

40. $3x+2y=5,\ 7x-6y=8.$

41. Find the fraction which becomes $\frac{1}{3}$ if 1 be added to its numerator, but, if 1 be added to its denominator, becomes $\frac{1}{4}$.

42. Find that number of two figures to which, if the number formed by changing the places of the digits be added, the sum is 121; and the difference of the same two numbers is 9.

43. A bill of £21. 10s. was paid with half-crowns and half-sovereigns, and the number of half-sovereigns exceeded three times the number of half-crowns by 4; how many were there of each?

44. A farmer sells to one person 9 horses and 7 cows for £300, and to another 6 horses and 13 cows for the same sum; find the price of a cow.

45. A purse contains £2. 12*s.* made up of pence, shillings, and half-crowns. The number of shillings is three times the number of half-crowns, but is only one-fourth the number of pence. Find the number of each coin.

46. A bill of £5 is paid partly in English shillings and partly in French francs. If the number of shillings were decreased by one-third, the number of francs would have to be increased by one-half. Find the number of each, assuming that twenty-five francs equal £1.

47. The garrison of a town has provisions for a certain number of days and it is known that, if a thousand men were to leave the garrison, the provisions would last two days longer but that, if a thousand more men were to join it, the provisions would last one day less. Find the number of the garrison.

48. A father is 30 years older than his son; in 20 years' time he will be double the son's age. Find the present age of the son.

49. A grocer buys eggs at 6*s.* 6*d.* per hundred. He sells all but 69 of them at the rate of 11 for a shilling, and then finds that he has received 30*s.* more than he gave for the whole number. How many eggs did he buy?

50. A bag contains a certain number of gold coins. Half its contents and half a sovereign more are removed. Half of what remains and half a sovereign more are then removed, leaving £5 in the bag. Find how much the bag contained originally.

51. A boy bought 104 apples for five shillings. The large ones he got at the rate of two for three-halfpence and the small ones two a penny. How many of each sort did he buy?

52. A train leaves London for Brighton at 9 a.m., travelling at the uniform rate of a miles an hour. Another train leaves Brighton for London at 10 a.m. and travels uniformly at b miles an hour. At what time will they meet, and at what distance from London, the distance between the two towns being taken at 50 miles?

53. At what time between 8 and 9 o'clock are the hands of a clock in the same straight line?

54. The dial of a clock shows 24 hours, so that the hour hand makes only one revolution per day and points straight upwards at midnight. At what time between 5 and 6 p.m. do the hands coincide?

Identities.

111. The two sides of an identity are really the same expression written in different forms, so that an identity is true for all values of x. It is often asked to prove that a given equality is an identity. This should be done by one of the following methods:

(i) Start with either side, and by removing brackets or by other algebraical processes transform it into the other side.

(ii) Take the two sides separately and prove them both equal to some third form.

Example. *Prove that* $a^2(b-c)+b^2(c-a)+c^2(a-b) \equiv bc(b-c)+ca(c-a)+ab(a-b)$.

Method I.
$$\begin{aligned} \text{L.H.S.} &= a^2b-a^2c+b^2c-b^2a+c^2a-c^2b \\ &= b^2c-bc^2+c^2a-ca^2+a^2b-ab^2 \\ &= bc(b-c)+ca(c-a)+ab(a-b) \\ &= \text{R.H.S.} \end{aligned}$$
and identity is proved.

Method II.
$$\begin{aligned} \text{L.H.S.} &= a^2b-a^2c+b^2c-b^2a+c^2a-c^2b \\ &= a^2(b-c)-a(b^2-c^2)+bc(b-c) \\ \text{R.H.S.} &= bc(b-c)+c^2a-a^2c+a^2b-ab^2 \\ &= a^2(b-c)-a(b^2-c^2)+bc(b-c). \end{aligned}$$
$\therefore$ L.H.S. = R.H.S.
and identity is proved.

The following very common method should **never be used**:

$a^2(b-c)+b^2(c-a)+c^2(a-b) = bc(b-c)+ca(c-a)+ab(a-b)$.

Remove brackets:

$a^2b-a^2c+b^2c-b^2a+c^2a-c^2b = b^2c-bc^2+c^2a-ca^2+a^2b-ab^2$.

Transpose and collect: $0 = 0$.

This method is bad: (i) it begins by assuming what has to be proved; (ii) it does not prove what was asked, but proves that $0 = 0$.

112. The following is a common type of example.

If possible, find values of A *and* B *so that*
$$A(x-2)+B(x-3) = 3x-5$$
shall be an identity.

Method I. The equality will be an identity if the coefficients of x on each side are the same, and if the constants are also the same.

$$A(x-2)+B(x-3) \equiv 3x-5,$$

if $\quad Ax-2A+Bx-3B \equiv 3x-5;$

i.e. if $\quad (A+B)x-(2A+3B) \equiv 3x-5;$

i.e. if $\quad A+B=3,$

and $\quad 2A+3B=5,$

solving $\quad A=4, \quad B=-1.$

Method II. If the equality is an identity it will be satisfied by any value of x we like to substitute.

$$A(x-2)+B(x-3) \equiv 3x-5.$$

Put $x=3$. $\quad A(3-2)+B(0) = 9-5;$

$$\therefore \quad A=4.$$

Put $x=2$. $\quad A(2-2)+B(2-3) = 6-5;$

$$\therefore \quad B=-1.$$

If what is asked is possible, these are the values.

Verify L.H.S. $= 4(x-2)-(x-3) = 3x-5 =$ R.H.S.

The Remainder Theorem.

113. In an Algebraic division sum the final remainder is of lower degree than the divisor; hence, if the divisor is of the first degree in x, the remainder is of no degree in x, i.e. the remainder does not contain x.

Suppose a function of x, viz. $f(x)$, is divided by $x-a$ and that the quotient is Q and the remainder R. Then Q will usually contain x, but R, as has just been shown, will not contain x.

$$\frac{f(x)}{x-a} = Q + \frac{R}{x-a}. \qquad \text{see § 79}$$

This statement is true for all values of x, that is, it is an identity.

Multiply by $x-a$, $\quad f(x) \equiv Q(x-a)+R.$

Being true for all values of x, this is true when $x=a$.

Put $x=a$, $\quad f(a) = 0+R$, since $x-a=0$.

Rewrite this last result, $\quad R=f(a)$ when $x=a$.

But since R does not contain x, it will have the same value whatever be the value of x.

$\therefore$ Whatever be the value of x, $R=f(a)$.

Hence when $f(x)$ is divided by $x-a$, the remainder is $f(a)$. This result is so important that it should be recollected in full:

When an expression containing x is divided by $x-a$, the remainder may be found without division by substituting a instead of x in the dividend.

The Factor Theorem.

114. If there is no remainder in a division sum, the divisor is a factor of the dividend. Hence it follows from the remainder theorem that, **if $f(a)=0$, $x-a$ is a factor of $f(x)$.**

Or, in full:

If, when a is substituted for x in an expression, the result is zero, then $x-a$ is a factor of that expression.

Suppose $x=a$ is a solution of the equation

$$f(x)=0,$$

then $$f(a)=0.$$

It follows from the Factor Theorem that if $x=a$ is a solution of $f(x)=0$, then $x-a$ is a factor of $f(x)$; and conversely, if $x-a$ is a factor of $f(x)$, then $x=a$ is a solution of $f(x)=0$.

Example I. *Find the remainder when x^4-2x^3+x-3 is divided by $x-2$.*

Here $f(x)\equiv x^4-2x^3+x-3$

$f(2)=2^4-2\,.\,2^3+2-3$

$=-1$;

$\therefore$ required remainder is -1.

Example II. *Are $x-2$ and $x+3$ factors of $x^3-7x^2+13x-6$?*

Here $f(x)\equiv x^3-7x^2+13x-6$,

$f(2)=2^3-7\,.\,2^2+13\,.\,2-6$

$=8-28+26-6$

$=34-34$

$=0$;

$\therefore$ $x-2$ is a factor.

$f(-3)=(-3)^3-7\,(-3)^2+13\,(-3)-6$

$=-27-63-39-6$

$\neq 0$;

$\therefore$ $x-(-3)$, i.e. $x+3$, is not a factor.

Examples XIII d.

Find, if possible, values of A, B, C, such that the following equations shall be identities; verify in each case.

1. $A(x-2)+B(x-3)=3x-8$.
2. $A(x-1)+B(x-1)(x-2)=3x^2-7x+4$.
3. $A(x+1)+B(2x-1)=x+4$.
4. $A(2x+1)+B(3x+2)=7x-13$.
5. $A(x-2)+B(x-3)+C(x-4)=3x-4$.
6. $x^2-2x+1=A(x^2+1)+B(x-2)+C$.
7. $(x-1)(x-2)=(Ax+B)(x-3)$.
8. $3x-4=A(2x-3)+B(3x-4)$.
9. $A(x-1)(x-2)+B(x-2)(x-3)+C(x-3)(x-1)=x$.
10. $A(x-1)(x-2)+B(x-2)+C=3x-2$.

Prove the following identities:

11. $(b-c)(x-a)+(c-a)(x-b)+(a-b)(x-c)\equiv 0$.
12. $(x-1)(x-2)+(x-2)(x-3)\equiv 2(x-2)^2$.
13. $(2x-3)^2-(3x-5)^2+(x-2)^2\equiv(2x-3)(4-2x)$.
14. $3(x-2)+4(x-3)\equiv(\frac{7}{2}x-9)^2-(\frac{7}{2}x-10)^2+1$.
15. $(x-1)^2-2(x-1)(x-2)+(x-2)^2\equiv 1$.

Without actual division find the remainder when

16. x^3-3x^2+x-1 is divided by $x-2$.
17. x^4+1 is divided by $x-3$.
18. x^3-5x+4 is divided by $x+1$.
19. $x^4-3x^3+2x^2-3x+1$ is divided by $x-2$.
20. x^5-2x^2-3 is divided by $x+3$.
21. $x^{50}-x+2$ is divided by $x+1$.
22. $x^{48}-2x^{47}+x^9-2x^8+x^3-x^2+4$ is divided by $x-2$.
23. $x^4-3x^3+2x^2-1$ is divided by $x-a$.
24. $x^7+6x^6-3x^3+4$ is divided by $x+p$.
25. y^3-3y^2+4y-a is divided by $y-x$.
26. Show that $x-1$ is a factor of
    (i) $x^4-3x^3+5x^2-4x+1$;
    (ii) $3x^3-x^2+2x-4$;
    (iii) $x^{18}-14x^{14}+13$.
27. If $f(x)=x^3+2x^2-5x-6$, find the values of
    $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$.

Hence find the factors of $f(x)$.

By finding the values of $f(x)$ for different values of x, find the factors of $f(x)$ when $f(x)\equiv$

28. x^3+x-2.
29. x^3-3x^2+2x.
30. x^3-4x^2+x+6.

Multiplication and Division.

115. The importance of mental work cannot be too strongly insisted upon. Such sums as $(x+5)(x-2)$, $\frac{x^2-8x+12}{x-2}$, should be worked mentally as explained in § 73 and § 80. Mental work is quite as accurate as written work, saves time and space, and looks neater.

The rules contained in the formula $(a+b)(a-b) = a^2-b^2$ are extremely useful both in Arithmetic and Algebra.

Example I.

$$\begin{aligned}&(3a-4b+5c)(3a+4b-5c)\\&=\{3a-(4b-5c)\}\{3a+(4b-5c)\}\\&=(3a)^2-(4b-5c)^2\\&=9a^2-16b^2+40bc-25c^2.\end{aligned}$$

Example II.

$$\begin{aligned}&(x^4-2x^3+3x^2-4)(x^4+2x^3+3x^2+4)\\&=\{(x^4+3x^2)-(2x^3+4)\}\{(x^4+3x^2)+(2x^3+4)\}\\&=(x^4+3x^2)^2-(2x^3+4)^2\\&=x^8+6x^6+9x^4-4x^6-16x^3-16\\&=x^8+2x^6+9x^4-16x^3-16.\end{aligned}$$

Notice that the terms whose signs are the same in both factors are placed in the first bracket, those whose signs are different in the two factors are placed in the second bracket. Thus in Algebra the formula is used to work out the product of a sum and difference as the difference of two squares; in Arithmetic it is more generally used the reverse way.

Thus, 4934^2-2562^2

$$\begin{aligned}&=(4934+2562)(4934-2562)\\&=7496\times2372\\&=17780512.\end{aligned}$$

$$\begin{array}{r}7496\\2372\\\hline14992\\22488\\52472\\14992\\\hline17780512\end{array}$$

If this had been worked by squaring, there would have been two long multiplication sums and one subtraction; whereas by this method there are one long multiplication, one subtraction, and one addition.

Detached coefficients.

116. If in the expression $5x^3+3x+4$ we put $x=10$, we obtain $5\times10^2+3\times10+4=534$. We see, therefore, that any arithmetical number can be obtained from an algebraical expression, arranged in descending powers of x, having as coefficients the respective digits that form the number. Hence all arithmetic sums can be worked in the same way as algebra sums, powers of 10 taking the place of powers of x.

Compare the two ways of finding the product of 237 and 569.

Arithmetic.	Algebra.
569	$5\times10^2+6\times10+9$
237	$2\times10^2+3\times10+7$
1138	$10^5+10^4+3\times10^3+8\times10^2$
1707	$10^4+7\times10^3+0\times10^2+7\times10$
3983	$3\times10^3+9\times10^2+8\times10+3$
134853	$10^5+3\times10^4+4\times10^3+8\times10^2+5\times10+3$

i. e. 134853.

It is seen that the Algebraic method is much longer owing to the continual repetition of the tens. The Arithmetic method shows that the index of 10 has been increased by one by placing the coefficient of the power in a column one step further to the left. Exactly the same thing can be done in Algebra, thus enabling us to omit the x's and to put down only the coefficients. It is advisable to write the powers of x at the top of the columns, just as we write £. s. d., or tons, cwt., qr., lb., in Arithmetic.

Multiply $5x^2+6x+9$ *by* $2x^2+3x+7$.

Long method.

$$\begin{array}{r} 5x^2+6x+9 \\ 2x^2+3x+7 \\ \hline 10x^4+12x^3+18x^2 \\ +15x^3+18x^2+27x \\ +35x^2+42x+63 \\ \hline 10x^4+27x^3+71x^2+69x+63 \end{array}$$

Short method.

x^4	x^3	x^2	x	
		5	+ 6	+ 9
		2	+ 3	+ 7
10	+12	+18		
	+15	+18	+27	
		+35	+42	+63
$10x^4$	$+27x^3$	$+71x^2$	$+69x$	$+63$

This short method is called the method of **Detached**

coefficients, and can be applied to addition, subtraction, multiplication, or division.

We will now give an example of the application of detached coefficients to Horner's method of long division.

Divide $5x^4-4x^3+3x^2+22x+55$ by x^2-3x+5.

	x^4	x^3	x^2	x	
1	5	− 4	+ 3	+22	+55
+3		+15	+33	+33	
−5			−25	−55	−55

$5x^2+11x+11$ Quotient.

Find the remainder when x^5-3x^2-5 is divided by $x-3$.

	x^5	x^4	x^3	x^2	x	
1	1			−3		−5
+3		+3	+9	+27	+72	+216

$x^4+3x^3+9x^2+24x+72$

Remainder 211.

Verify by remainder theorem.

Remainder $= 3^5-3^2-5$
$= 243-27-5$
$= 211$

Note. It is often found that the easiest way of calculating $f(a)$ is to find by Horner's method the remainder when $f(x)$ is divided by $x-a$; thus reversing the use of the remainder theorem.

Examples XIII e.

1. Multiply $2a-3b+c$ by $2a+3b+c$.
2. Find the product of $3x^2-2x-1$ and $3x^2+2x+1$.
3. Find as shortly as possible the product of
   x^3+2x^2-x+2 and x^3-2x^2+x+2.
4. Simplify $(x^3+x^2-3x-4)^2-(x^3-x^2-3x-4)^2$.
5. Calculate as shortly as possible 9987^2-9887^2.
6. Find the area of a circular racing track of which the outer radius is 259 ft. and the inner radius 238 ft. ($\pi = \frac{22}{7}$).

In the following examples use the method of detached coefficients:

7. Multiply x^3+2x^2+4x+8 by $x+2$.
8. Multiply x^3+2x^2+3x+4 by x^3-3x+2.

9. Multiply $3x^3+3x^2-x+2$ by $2x^3-3x+1$.

10. Add $3x^2-2ax+5a^2$, $3ax-7a^2+5x^2$, $3a^2-7x^2-4ax$; and subtract $x^2-4ax+a^2$ from the sum.

11. Find the sum of $5x^2-3x-2a$, $4x+3a-3x^2$, $a-2x^2-5x$.

12. Divide $6x^4-19x^3+15x^2+8x-12$ by $2x-3$.

13. Divide $2x^6-3x^5-9x^4+48x+16$ by x^2-4.

14. Multiply x^3-3x^2+3x-1 by $x^2-8x+12$.

15. Divide $2x^5-6x^4+x^3+7x^2-4$ by x^3-3x+2.

16. Divide $x^4+x^3-5x^2+18x-8$ by x^2+3x-2.

17. Divide $6x^5-11x^3+20x^2+23x-38$ by $2x^2-4x+5$.

18. Divide $x^4+18x^2-216x-3888$ by $x^2+12x-108$.

19. Multiply $x^4-2x^3+4x^2-8x+16$ by $x+2$.

20. Divide x^6-6 by x^3-1.

XIII. Revision Examples.

1. Add together $2x+3y-4z$, $5z+5x-4y$, and $7y-3x-8z$; from the sum of these subtract $2x-y-6z$.

2. Multiply x^3+3x+2 by $x-3$, and prove the answer.

3. If $x=3$, $y=0$, $z=-2$, find the value of

(i) $\sqrt{x^2+y^2+4z^2}-\dfrac{x}{z}-\dfrac{8x^3}{z^3}$;

(ii) $z^x-\sqrt[5]{x^3-xz+3z^2}$.

4. Prove that if $a>b>c>0$ then $a-(b-c)=a-b+c$.

5. Solve $\frac{1}{2}(9-2x)=\frac{3}{4}-\frac{1}{10}(7x-18)$.

6. A post is a quarter of its length in the ground, a third of its length in water, and rises 10 feet above the water; what is its whole length?

---

7. Compare the meaning of 59 in Arithmetic with that of ab in Algebra; also the meaning of $5\frac{3}{8}$ and $a\dfrac{b}{c}$.

8. Find the value of $\dfrac{3a^2b^2c}{4a(3bc-4a)}\times(ab+bc+ac)$ when $a=3$, $b=-2$, $c=-4$.

9. Divide $6a^4-13a^3b-4a^2b^2+5ab^3-4b^4$ by $\dfrac{2a^2-3ab-4b^2}{5}$.

10. Show that, if unity be added to the product of any four consecutive numbers, the sum is a perfect square.

11. Solve $\frac{1}{5}(\frac{2}{3}x+4)+\frac{2x-15}{18}=\frac{x}{6}(\frac{6}{x}-1)$.

12. A is twice as old as B; twenty-two years ago he was three times as old. What is A's present age?

---

13. Multiply $ax-(a-2)$ by $ax-(a+2)$.

14. What should be the price of 7 eggs when a dozen cost x pence?

15. Simplify $1-[2+\{3-(1+2)-3\}+1]-2$.

16. Simplify $\frac{a-2}{2}-\frac{a-3}{3}+\frac{a-4}{4}-\frac{a-6}{6}$.

17. Solve (i) $\frac{7x}{3}-\frac{5y}{4}=2\frac{1}{6}$, $4x=3y$;

(ii) $\frac{x-1}{2}+\frac{3x-1}{4}-\frac{5x-1}{6}=\frac{5x-7}{12}$.

18. What is the meaning of $f(x)$?

If $f(x)=x^3-3x+2$, what are the values of $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$?

Draw the graph of $y=x^3-3x+2$ from $x=-1$ to $x=+6$.

---

19. When $a=1$, $b=5$, $c=-1$, find the value of

(i) $(b+c)(c+a)(a+b)$;

(ii) $3abc-a^3-b^3-c^3$.

20. Multiply $x^3+4y^3+9z^3+6yz+3zx-2xy$ by $x+2y-3z$.

21. Find the square root of

$$x^6-6x^4y^3+8x^3y^3+9x^2y^4-24xy^5+16y^6.$$

22. Solve $x-y=3(x+15)$,

$x+y=5(y-9)$.

Verify by a graphical method.

23. A square of carpet is cut up into strips so as to cover a border $(a+b)$ feet wide round the floor of a room whose length is $10a$ feet and breadth $10b$ feet. What was the length of the side of the square of carpet?

24. Prove that $f(a)$ is the remainder when $f(x)$ is divided by $x-a$.

---

25. Subtract $(1+2x)(1+2x-x^2)(1-2x)(1-2x-x^2)$ from the product of $1-2x(1-2x+x^2)$ into $1+2x(1+2x+x^2)$.

26. Write down an expression for the number of which the first digit is a and the second digit b.

Write down also the six numbers that can be formed with the three digits a, b, c in various positions, and find the sum of these six.

27. If a and b be any positive integers, prove that $ab = ba$ and $a+b = b+a$.

28. Given $2y = 5x-6$, find the value of y when $x = 3{\cdot}2$ and when $x = -2{\cdot}4$.

Hence represent graphically the equation $2y = 5x-6$.

What will the equation $2y = 5x-8$ represent?

29. Solve $\left.\begin{aligned}(x-4)(y-3) &= xy-84\\ 6x-5y-27 &= 0\end{aligned}\right\}$.

30. A sum of £5 is divided between two men so that one receives eight times as many pence as the other shillings. What sum does each receive?

---

31. Subtract $\dfrac{3x-6y}{3}+a$ from $x-\dfrac{4y-2c}{2}$.

32. Prove that the square of the sum of any two numbers together with the square of their difference is equal to twice the sum of the squares of the two numbers.

33. Extract the square root of

$$4x^6-12x^5y+17x^4y^2-12x^3y^3+4x^2y^4.$$

34. Solve $\dfrac{x-b}{a} = y-b = \dfrac{x-a}{b}$.

35. Divide $x^5-x^4+8x^3-14x+15$ by x^2-x+3.

36. A man has five sons whose united ages are just equal to his own. In twelve years' time the united ages of the sons will be double that of their father. What is the father's age now?

---

37. Multiply $a^4-3a^3b+4a^2b^2-18ab^3-12b^4$ by $a^2+3ab-2b^2$.

38. Add together $5(a-2b)\times b\{3-5(c-2)\}$, $5(b-2)(c-3)$, and $(a+8)(c-4)-(a-2)(c+1)$.

39. A person who has c shillings buys a things each costing b pence; how many pence will he have left?

40. Solve $\dfrac{x-4}{5}+\dfrac{3-x}{6} = \dfrac{5-x}{30}$.

41. What is meant by the graph of a function? Plot the points $(-2, 6)$, $(1, -3)$ and draw the line which joins them. Find its equation.

42. I bought some engravings at 35*s.* each and some books at 16*s.* each. The total cost was £16. 15*s.*, and the number of books was 5 more than the number of engravings. How many were there of each?

---

43. Prove the identity $(ac-bd)^2+(ad+bc)=(a^2+b^2)(c^2+d^2)$.

44. If 1 be divided into any two parts, prove that the results of adding each part to the square of the other are equal.

45. Solve $\cdot 15x+\cdot 2-\cdot 875x+\cdot 375=\cdot 0625x-1$.

46. *A* and *B* begin to play with equal sums, and when *A* has won 30*s.* his money is $\frac{13}{7}$ of *B*'s; find how much each started with.

47. Multiply $x^2+4y^2+z^2-2yz+zx+2xy$ by $x-2y-z$.

48. If 8 men do a piece of work in 10 days working x hours a day, and 12 men do the same amount in 8 days working y hours a day, prove that $5x=6y$.

---

49. Divide $x^4-\frac{19}{8}x^2y^2+\frac{1}{3}xy^3+\frac{1}{9}y^4$ by $x^2+2xy+\frac{1}{3}y^2$.

50. Find the values of $(3ax^2z^3)^2$, $(-5a^3bx^3)^3$, $\left(-\frac{4p^3q}{3r^2}\right)^4$.

51. Solve $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=10a+11b$.

52. Solve $$\frac{3x+4y+3}{10}-\frac{2x+7-y}{15}=5+\frac{y-8}{5}.$$
$$\frac{9y+5x-8}{12}-\frac{x+y}{4}=\frac{7x+6}{11}.$$

53. A messenger starts on an errand at the rate of 4 miles an hour; another is sent $1\frac{1}{2}$ hours afterwards to overtake him; the latter walks at the rate of $4\frac{3}{4}$ miles an hour. When will he overtake the first messenger?

54. Prove the following rule:

To find the price of an oz. avoirdupois when the price per pound in shillings is known. Deduct one-fourth from the price in shillings and the result gives the answer in pence.

---

55. When $a=4$, $b=\frac{1}{2}$, $c=1$, find the value of
$$(a^2+b^2-c^2+2ab)\div(a^2-b^2-c^2+2bc).$$

56. Multiply $\frac{5}{2}x^2+3ax-\frac{7}{3}a^2$ by $2x^2-ax-\frac{1}{2}a^2$.

57. Find the square root of $1+2x+7x^2+6x^3+9x^4$.

58. Solve $\frac{1}{x}+\frac{1}{2x}-\frac{1}{8x}=\frac{7}{8}$.

59. Solve graphically and algebraically:

$$4x-2y=1{\cdot}2,$$
$$\cdot02x+3{\cdot}4y=-{\cdot}01.$$

60. The price y in pence of a dish x inches in diameter is given in the following table:

x	10	11	12	13	14	15	16	17
y	$14\frac{1}{4}$	$15\frac{1}{2}$	$16\frac{3}{4}$	$18\frac{1}{2}$	$23\frac{1}{2}$	$30\frac{1}{2}$		51

Draw a graph and obtain an approximate price for the 16 in. dish.

---

61. Prove the following rule: If the square of half the difference of two numbers is subtracted from the square of half the sum, the difference is the product of the numbers.

62. Divide $x^{10}-123x^5+1$ by x^2-3x+1.

63. If $a+b=x$ and $a-b=y$, express a^3-b^3 in terms of x and y. Hence find the value of 5002^3-4998^3.

64. Solve (i) $x+y+39=3x-y+43=2y-x+48$;

(ii) $\frac{1}{3}(x+1)+\frac{1}{4}(5-2x)-\frac{1}{2}(2+5x)+\frac{1}{6}(10-2x)=0$.

65. A greengrocer has a certain number of apples. He sells one-third of them at three for twopence, two-thirds of the remainder at ten for sixpence, and the rest go bad. On the whole he receives twenty-two shillings. How many apples had he?

66. Draw the graphs $3x-y=1$, $x+y=7$. Hence solve the equations and verify by calculation.

---

67. Simplify

$$(a+b)(a^2-b^2+c^2)+(c-a)(b^2-a^2-c^2)+(a+b-c)(b^2-c^2-a^2).$$

68. Find the value of $\dfrac{1}{y-z}+\dfrac{1}{z-x}+\dfrac{1}{x-y}$

when $x=4{\cdot}6$, $y=3{\cdot}4$, $z=2{\cdot}2$.

69. Solve (i) $9(x-9)=^1(x-9)-(x-9)$;

(ii) $\dfrac{2x-8}{15}-\dfrac{4x-9}{20}=\dfrac{27-5\frac{1}{2}x}{8}-\dfrac{9}{40}$.

70. Find the square root of $9x^4-6x^3-11x^2+4x+4$.

Find correct to two places of decimals the value of $x+y+xy$ when $x=\sqrt{2}$, $y=\sqrt{3}$.

71. What is the distinction in meaning between $a-(b-c)$ and $(a-b)+c$? Prove that they are equal.

72. In a physical experiment the times a body took to fall through certain distances were noted. Plot the following observations on squared paper and draw the graph:

Height of fall, ft.	4·5	4	3	2	1	·5	2·5
Time of fall, secs.	15	14·1	12·2	10	7·1	5	11·2

CHAPTER XIV

FACTORS

117. Facility in resolving expressions into factors is so important that we repeat in this chapter the various hints and rules that have already been given. All examples on resolving into factors should be mentally verified; in fact the methods for mental multiplication and for resolving into factors should be studied together.

118. Case I. Whenever all the terms of an expression contain the same factor that factor can be placed outside a bracket, and the quotient obtained by dividing by that factor must be placed inside the bracket.

Examples of this are given in Examples V b.

Harder Example. *Find two factors of*

$$(x+a)\,x^2-(x+a)\,(a+b)\,x+(x+a)\,ab.$$

$$\text{Expression} = (x+a)\,x^2-(x+a)\,(a+b)\,x+(x+a)\,ab$$
$$= (x+a)\{x^2-(a+b)x+ab\}.$$

119. Case II. It often happens that one factor will divide part of an expression and another factor the rest of the expression, the two quotients being the same. When this is the case, the factors of the whole expression can be found as shown in the following example.

Example I. *Factorise* $ax^2+2by^2+2bx^2+ay^2$.

It is seen that a divides two of the terms and $2b$ the other two.

Expression $= ax^2 + ay^2 + 2bx^2 + 2by^2$
$= a(x^2+y^2) + 2b(x^2+y^2)$
$= (x^2+y^2)(a+2b).$

At once verify mentally to avoid making mistakes.

Example II. *Find the factors of* $x^3 - 3x^2 - x + 3.$

Expression $= x^3 - x - 3x^2 + 3$
$= x(x^2-1) - 3(x^2-1)$
$= (x-3)(x^2-1).$

This could also have been arranged as follows:

Expression $= x^3 - 3x^2 - x + 3$
$= x^2(x-3) - (x-3)$
$= (x-3)(x^2-1).$

Examples XIV a.

Resolve each of the following expressions into as many factors as possible:

1. $ax^2 - 2a^2x + 2a^3.$
2. $3x^2y - 4xy^2 + 5xyz.$
3. $6px - 3py + 6pq.$
4. $8ab^2 - 10abc + 12bx^2.$
5. $5x^2 - 15xy + 20y^2.$
6. $x(x-3) + y(x-3) - x + 3.$
7. $7a^2(a-1) - 3ab(a-1) + b^2(1-a).$
8. $xya^2 - 2xyab + xyb^2.$
9. $a^3 - 3a^2 + ay - 3y - a + 3.$
10. $x^2(a+b+c) - xy(a+b+c) + y^2(a+b+c).$
11. $25x^4y - 40x^2y^2 + 16x^3y^3.$
12. $ac + ab - bc - c^2.$
13. $4(x-2)^2(x-3) - (x-2)^3.$
14. $a^2 + b^2 + 2(ab + ac + bc).$
15. $ab(c^2+d^2) + cd(a^2+b^2).$
16. $a^2 - 2ab + b^2 - a + b.$
17. $x^2 - a + ax - x.$
18. $a^2 + ab + ac + bc.$
19. $2ax + 3bx - 4ay - 6by.$
20. $6mp - 9mq - 4pn + 6qn.$

By mental work only find the product of:

21. $x-4$ and $x+2.$
22. $x+3$ and $x+5.$
23. $x-2y$ and $x-4y.$
24. $a-3b$ and $a+2b.$
25. $a+2b$ and $a-2b.$
26. $3x+4y$ and $3x-4y.$
27. $a+b+1$ and $a+b-1.$
28. $9p+10q$ and $9p-10q.$
29. $3k-5l$ and $3k+5l.$
30. $x+y+a+b$ and $x+y-a-b.$

120. Case III. By actual multiplication it can be shown that

$$(a+b)(a-b) = a^2 - b^2.$$

This is one of the most important formulae in Algebra and the two corresponding rules should be learnt thoroughly.

The product of the sum and difference of two quantities is equal to the square of the first minus the square of the second.

The difference of the squares of two quantities is equal to the first plus the second multiplied by the first minus the second.

It should be noticed that the quantities may be numbers, single terms or multinomials (§ 94). In applying these rules, the formulae for squaring binomial and multinomial expressions will be continually used, and should, therefore, be revised (§§ 87, 91).

Note. Binomials of the form x^2+y^2 have no factors.

Examples XIV b.

Find the factors of:

1. $4x^2-y^2$.
2. $9a^2-16b^2$.
3. $16p^2-25q^2$.
4. $9x^2-25a^2$.
5. $64a^2-81x^2$.
6. $49x^2y^2-36a^2b^2$.
7. p^2-121q^2.
8. $10p^2-40x^2$.
9. $12a^2-75b^2$.
10. 100^2-2^2.
11. 79^2-21^2.
12. 84^2-68^2.
13. $(x+y)^2$.
14. $(a-b)^2-x^2$.
15. $(2a-3b)^2-4a^2$.
16. $(3x+4y)^2-16y^2$.
17. $(3a-6b)^2-81c^2$.
18. $(2a+3b)^2-(a-2b)^2$.
19. $(4x-5y)^2-9z^2$.
20. $(2x-3a)^2-(a-2x)^2$.

Find the product of:

21. $x+2y$ and $x-2y$.
22. $2x-3y$ and $2x+3y$.
23. $x+y+1$ and $x+y-1$.
24. $2x-3y-1$ and $2x+3y-1$.
25. $a+b+c$ and $a-b-c$.
26. $2a-b+3c$ and $2a+b+3c$.
27. $a+b+c-d$ and $a-b+c+d$.
28. 198 and 202.
29. 91 and 109.
30. 78 and 82.

Factorize:

31. $a^2+2ab+b^2-c^2$.
32. $4a^2+4ab+b^2-9x^2$.
33. $a^2-2ac+c^2-b^2+2bd-d^2$.
34. $x^2+2xy+y^2+1$.
35. $a^2+6ab+9b^2$.
36. $ax^2+2axy+ay^2-a^3$.
37. $a^2-3a-6b-4b^2$.
38. x^2+x-y^2+y.

39. $a^2-b^2-c^2+d^2-2ad+2bc$.
40. $4x^2-9y^2+25z^2-20xz$.
41. $2x(2x+a)-y(y+a)$.
42. $1+2ab-a^2-b^2$.
43. $1-3x-3xy-y^2$.
44. $a^2+2bc-(c^2+2ab)$.
45. x^3-4x.
46. x^4-x^2.

Write down the quotients in the following cases:

47. $(x^2-a^2)\div(x+a)$.
48. $(1-p^2)\div(1+p)$.
49. $(3x^2-3y^2)\div(x-y)$.
50. $(10x^2-10b^2)\div 2(x-b)$.
51. $(27x^2-12y^2)\div(3x+2y)$.
52. $(16x^2-25a^2)\div(4x-5a)$.
53. $\dfrac{25a^2-49b^2}{5a+7b}$.
54. $\dfrac{72p^2-50q^2}{12p-10q}$.
55. $\dfrac{16-49a^2}{7a-4}$.
56. $\dfrac{16-49a^2}{4-7a}$.
57. $\{a^2-(b-c)^2\}\div(a+b-c)$.
58. $\{9a^2-16(6-c)^2\}\div(3a+46-4c)$.
59. $\{(x+2y)^2-(a-3b)\}\div(x+2y-a+3b)$.
60. $\dfrac{(1-x)^2-(3+2x)^2}{4+x}$.

Solve:

61. $(x-a)(a-x)=(b-x)(x-b)$.
62. $a(a-x)=b(x+b)$.
63. $x(a-x)-a^2=x(b-x)-b^2$.
64. $a(a+x)=b(b-x)$.
65. $x(2-x)-b^2=x(b-x)-4$.

121. Case IV. First method. We have seen in § 39 that

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

which is the same as

$$x^2+(a+b)x+ab=(x+a)(x+b).$$

In applying this it must be remembered that a and b need not be positive, and therefore either $a+b$ or ab or both may be negative. The four possible forms are shown by

$$x^2+6x+8=(x+4)(x+2),$$
$$x^2-6x+8=(x-4)(x-2),$$
$$x^2+6x-16=(x+8)(x-2),$$
$$x^2-6x-16=(x-8)(x+2).$$

In all four cases we have to find two numbers whose

product equals the constant term (the term that does not contain x, sometimes called the third term) and whose algebraic sum equals the coefficient of the first power of x.

Careful attention must be paid to the laws of signs. Thus if the third term is positive, the numbers are both positive or both negative;

if the third term is negative, the numbers are of unlike signs;

if the second term is positive, the numerically greater number is positive;

if the second term is negative, the numerically greater number is negative.

Examples XIV c.

Find the factors of:

1. x^2+5x+6.
2. x^2+6x+5.
3. x^2-6x+5.
4. x^2+6x+9.
5. $a^2-7ab+10b^2$.
6. $a^2+7a+12$.
7. $p^2+9p+18$.
8. $y^2-12y+32$.
9. $y^2+12y+35$.
10. x^2-6x+9.
11. $y^2-7y+10$.
12. a^2-6a+5.
13. x^2-6x-7.
14. $y^2-7xy-18x^2$.
15. $a^2-6a-108$.
16. $x^2-10xy+16y^2$.
17. x^2+9x+8.
18. $x^2+3x-10$.
19. $x^2+5x-14$.
20. $a^2+ab-2b^2$.
21. a^2-16b^2.
22. y^2-6y.
23. $x^2-9xy-10y^2$.
24. $12-4x+x^2$.
25. $12-4x-x^2$.
26. $x^2-16x+48$.
27. $x^2-13xy+36y^2$.
28. $a^2-20a+36$.
29. $a^2+12a+36$.
30. $a^2-37a+36$.
31. $a^2-5ax-36x^2$.
32. $y^2+9y-36$.
33. $y^2-16y+64$.
34. $x^2-35y-36$.
35. $y^2+5y-36$.
36. p^2-36.
37. $x^2-(p-q)x-pq$.
38. a^2-a-42.
39. a^2+a-56.
40. x^2-x-72.
41. $x^2-xy-132y^2$.
42. $x^2-x-600$.
43. x^2-x-12.
44. x^2+x-20.
45. $1+3x-2x^2$.
46. x^3-3x^2+2x.
47. $x^2+2x-15$.
48. $x^2-7x-30$.
49. k^2-3k+2.
50. $2x^2-ax-a^2$.
51. $(x-2)^2+x^2-5x+6$.
52. $(x-3)^2+(x-3)(x-4)$.
53. $(x+4)+x^2-3x-28$.
54. $16-6x-x^2$.

Simplify the following fractions:

55. $\dfrac{x^2-5x+6}{x-2}$.
56. $\dfrac{a^2-17a+72}{a-9}$.

57. $\dfrac{a^2-17a+72}{a^2-14a+45}$.

58. $\dfrac{x^2-ax-56a^2}{x+7a}$.

59. $\dfrac{y^2-16xy-17x^2}{17x-y}$.

60. $\dfrac{y^2-24xy+63x^2}{y^2+15xy-54x^2}$.

122. Case IV. Second method. If the constant term is large it is not easy to find the factors; in such cases the following method may be used.

$$\begin{aligned} x^2-2x-63 &= x^2-2x+1-64 \\ &= (x-1)^2-8^2 \\ &= (x-1+8)(x-1-8) \\ &= (x+7)(x-9). \end{aligned}$$

It is seen that a number is added which, together with the first two terms, will make an exact square; in order that the expression may be unaltered in value, the number added must also be subtracted; if the resulting number is also an exact square, the original expression is reduced to the difference of two squares and can be factorized by Case III.

123. Case V. First method. To factorize expression of the form ax^2+bx+c.

We have to find two expressions, $px+q$ and $rx+s$, such that

$$(px+q)(rx+s) \equiv ax^2+bx+c.$$

Comparing the two sides it is seen that

$$\begin{aligned} pr &= a, \\ qs &= c, \\ ps+rq &= b. \end{aligned}$$

Examples. *To factorize* $20x^2+9x-18$.

From the earlier part of this section we know that we have to find two numbers whose product is 20 and two others whose product is -18, such that the sum of the cross products is 9.

The factors of 20 are 20×1, 10×2, 5×4.

The factors of 18 are 18×1, 9×2, 6×3.

Since the third term is -18 the constant terms in the factors must be of different signs, so that the algebraical sum of the cross products is an actual difference.

$(20x+18)(x-1)$ gives -2 as coefficient of x, (i)

$(20x+1)(x-18)$ gives -359 as coefficient of x, (ii)

$(10x-9)(2x+2)$ gives $+2$ as coefficient of x, (iii)
$(5x+6)(4x-3)$ gives $+9$ as coefficient of x; (iv)
and these are therefore the required factors.

There are nine possible arrangements of the factors of 20 and 18, and allowing for the differences of sign there are eighteen possible coefficients of x. But it is not necessary to try all these, as combinations like i and iii are obviously not the correct factors; for i has a numerical factor 2 and iii has also a factor 2, while the original expression has not got a numerical factor (if it had it should be removed at once and put outside a bracket as in Case I). Also number ii and similar arrangements need not be considered as they obviously give too big a coefficient for x.

The actual work is done mentally, but the working of the correct pair should be put down at the side to show how the factors are obtained.

The work should be set down somewhat as follows:

$$20x^2+9x-18=(5x+6)(4x-3) \qquad \begin{matrix} 20\times1 & 10\times2 & 5\times4 \\ 18\times1 & 9\times2 & 6\times3 \\ 6\times4 & -5\times3 & =9. \end{matrix}$$

This method is complicated when the coefficient of x^2 and the third term have several pairs of factors, as in the example worked above. When they each have only one or two pairs of factors, or when they involve letters and not numbers, the method is quite simple.

$$39x^2+371x-58=(13x-2)(3x+29) \qquad \begin{matrix} 39\times1 & 13\times3 \\ 58\times1 & 29\times2 \\ 13\times9- & 2\times3 & =371. \end{matrix}$$

$$abx^2+(a^2-b^2)x-ab=(ax-b)(bx+a) \qquad \begin{matrix} a\times b \\ b\times a. \end{matrix}$$

124. Case V. Second method. With numerical coefficients the following method often takes less time as it does not require so much trial work:

$$20x^2+9x-18=\frac{20^2x^2-9\times20x-18\times20}{20} \qquad \begin{matrix} 18\times 20 \\ 6\times 60 \\ 12\times 30 \\ 3\times120 \\ 15\times 24 \end{matrix}$$

$$=\frac{(20x+15)(20x-24)}{20}$$

$$=(4x+3)(5x-6).$$

Step 1. Multiply and divide by the coefficient of x^2.

Step 2. Treat $20x$ as a single quantity; we then have Case iv, since the numerator is of the form $y^2 - 9y - 18 \times 20$. Find factors of 18×20 whose difference is 9; the trial work is shown at the side.

Step 3. Cancel any factors common to the numerator and denominator. It will be found that the denominator always reduces to unity.

In the following examples expressions of the form ax^2+bx+c should at first be factorized by both methods. Experience will teach which is the better method to use in particular cases.

Examples XIV d.

Find the factors of:

1. $x^2-42x+432$.
2. $x^2-16x-377$.
3. $x^2+12x-748$.
4. $9x^2-27x-2736$.
5. $2x^2-x-3$.
6. $ap^2-apq-72aq^2$.
7. $36x^2-42x+12$.
8. $12a^2+7ab-12b^2$.
9. $5y^2-2y-7$.
10. $10a^2-9a-7$.
11. $10x^2-13x-9$.
12. $15a^2+32a+9$.
13. $4x^2-15xy-4y^2$.
14. $18x^2+53x-35$.
15. $12x^2+11x-15$.
16. $2x^2+3x-35$.
17. $2x^2+x-15$.
18. $3x^2-5x-2$.
19. $3x^2+5xy-2y^2$.
20. $12x^2-x-20$.
21. $14x^2-93x+13$.
22. $6x^2-49x+65$.
23. $8x^2-22x+15$.
24. $abx-(a^2-b^2)x+ab$.
25. $4abx-6(a^2-b^2)x+9ab$.
26. $6apx^2-(4a^2-9p^2)^2xy-6apy$.
27. $ax^2-(3ab+2)x+6b$.
28. $6a^2x^2-5abx-2b^2$.
29. $10pqx^2-(5p^2+2q^2)x+pq$.
30. $6a^2x^2-15axy+3y^2$.
31. $9x^2+14x-8$.
32. $30x^2-45x-30$.
33. $ax^2-(9a+2)x+18$.
34. $12ax^2-(4a^2-36)xy-12ay^2$.
35. $4x^2+x-3$.
36. $6x^2-19x+15$.
37. $27x^2-48x+16$.
38. $6x^2-7xy-5y^2$.
39. $p(x^2-1)-x(p^2-1)$.
40. $2a^2+3ab-2b^2$.
41. $6x^2-5x-6$.
42. $6x^2-7x-20$.
43. $ab(x^2+y^2)+xy(a^2+b^2)$.
44. $8x^2+13x-6$.
45. $16x^2-24x+9$.
46. $12x^2-23x-24$.
47. $(a^2-1)x^2+2(a^2+1)x+a^2-1$.
48. $(a^2-1)x^2-4ax+1-a^2$.
49. $(a^2+1)(x^2-1)-a^2(a^2+2)x$.
50. $(a^2+1)x^2+a^4x+a^2-1$.

125. Case VI. To resolve the sum or difference of two cubes into factors.

It was shown in § 74 by actual multiplication that

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

and $$a^3+b^3=(a+b)(a^2-ab+b^2);$$

or in words—

The difference of the cubes of two quantities equals the first quantity minus the second multiplied by the sum of the squares of the quantities increased by their product.

The sum of the cubes of two quantities equals the sum of the two quantities multiplied by the sum of the squares of the quantities diminished by their product.

As in the case of the difference of two squares the quantities may be numbers, single algebraical terms, or multinomial expressions, e.g.

$$512-125x^3=8^3-(5x)^3$$
$$=(8-5x)(64+40x+25x^2).$$

$$(a+b)^3+(x+y)^3=(\overline{a-b}+\overline{x-y})\{(a+b)^2-(a+b)(x+y)+(x+y)^2\}.$$

126. Case VII. The division sum worked at the end of § 82 provides the means for resolving into factors expressions of the form $x^3+y^3+z^3-3xyz$.

It is there shown that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$
or $$(x+y+z)(x^2-xy-xz+y^2-yz+z^2).$$

In using this formula or any other formula containing the third power the student must recollect that $(-y)^3=-y^3$.

Find the factors of $x^3-8y^3+27+18xy$.

Expression $=x^3+(-2y)^3+3^3-3x(-2y)\times 3$
$=(x-2y+3)(x^2+2xy-3x+4y^2+6y+9).$

It is worthy of notice that

$$x^2+y^2+z^2-yz-zx-xy=\tfrac{1}{2}\{(y-z)^2+(z-x)^2+(x-y)^2\}.$$

Examples XIV e.

Find the factors of:

1. $8-64y^3$.
2. $1-216x^3$.
3. $125+8x^3$.
4. $8a^3+27b^3$.
5. $343-x^3y^3$.
6. x^3-8.

7. $108a^3-500$.
8. $27y^3+1000z^3$.
9. $a^3-b^3+c^3+3abc$.
10. $a^3-b^3-c^3-3abc$.
11. $a^3-b^3+1+3abc$.
12. $a^3+b^3-1-3ab$.
13. x^3+8y^3.
14. $24a^3-3b^3$.
15. $8x^3-y^3$.
16. $16x^3-54y^3$.
17. $27x^3-64y^3$.
18. $64x^3-125$.
19. $8x^3+27y^3-1+18xy$.
20. $64-8x^3+125y^3+120xy$.
21. a^3-64ab^3.
22. $x^3+y^3+3xy(x+y)$.
23. $a^3x^3-64b^3x^3$.
24. $3pqr-p^3-q^3-r^3$.

What is the quotient when

25. $x^3+3xy+y^3-1$ is divided by $x+y-1$?
26. $a^3-6ab-b^3-8$ is divided by $a-b-2$?
27. $8x^3+125y^3-8+60xy$ is divided by $2x+5y-2$?
28. What is the value of $x^3+3xyz+y^3-z^3$ when $x=42$, $y=61$, and $z=-103$?
29. Divide $2(x^3+y^3+z^3-3xyz)$ by $(y-z)^2+(z-x)^2+(x-y)^2$.
30. If $2x+3y=5z$, prove that $8x^3+27y^3-125z^3+90xyz=0$.
31. Find the factors of $(x-y)^3+(y-z)^3$.
32. What value of x will make $(x-7)^3+(x-3)^3$ equal to nothing?
33. Show that $(2a+3b-4c)^3-(3a-2b+c)^3$ is equal to nothing if $a-5b+5c=0$.

Expressions of higher degree than the second.

127. There is no general rule for resolving into factors expressions which contain the third or higher powers. The methods already mentioned are often effective, and the following examples show how they can be used.

By using the facts that $x^4=(x^2)^2$, $x^6=(x^3)^2$, &c., expressions of higher degree can often be reduced to one of the forms already given.

$$\begin{aligned} x^4-y^4 &= (x^2)^2-(y^2)^2 \\ &= (x^2+y^2)(x^2-y^2) && \text{Case III.} \\ &= (x^2+y^2)(x+y)(x-y). \\ x^4+7x^2-18 &= (x^2)^2+7(x^2)-18 \\ &= (x^2+9)(x^2-2). && \text{Case IV.} \\ x^4+x^2y^2+y^4 &= x^4+2x^2y^2+y^4-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 && \text{Case III.} \\ &= (x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

This result is important, and it is advisable to recollect it as a separate formula.

$$\begin{aligned} x^4-17x^2y^2+16y^4 &= x^4+8x^2y^2+16y^4-25x^2y^2 \\ &= (x^2+4y^2)^2-(5xy)^2 \\ &= (x^2-5xy+4y^2)(x^2+5xy+4y^2) \qquad \text{Case III.} \\ &= (x-y)(x-4y)(x+y)(x+4y). \qquad \text{Case IV.} \end{aligned}$$

This last example could have been resolved rather more easily by the method for Case IV :

$$\begin{aligned} x^4-17x^2y^2+16y^4 &= (x^2)^2-17(x^2)(y^2)+16(y^2)^2 \\ &= (x^2-y^2)(x^2-16y^2) \\ &= (x-y)(x+y)(x-4y)(x+4y). \end{aligned}$$

$$\begin{aligned} x^3-3x^2y-3xy^2+y^3 &= x^3+y^3-3x^2y-3xy^2 \\ &= (x+y)(x^2-xy+y^2)-3xy(x+y) \qquad \text{Case II.} \\ &= (x+y)(x^2-xy+y^2-3xy) \\ &= (x+y)(x^2-4xy+y^2). \end{aligned}$$

Notice the resemblance of this expression to the expanded forms of $(x+y)^3$ and $(x-y)^3$, and also notice the differences.

$$\begin{aligned} x^6-y^6 &= (x^3)^2-(y^3)^2 \\ &= (x^3+y^3)(x^3-y^3) \qquad \text{Case III.} \\ &= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2); \qquad \text{Case VI.} \end{aligned}$$

or

$$\begin{aligned} x^6-y^6 &= (x^2)^3-(y^2)^3 \\ &= (x^2-y^2)(x^4+x^2y^2+y^4) \qquad \text{Case VI.} \\ &= (x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

128. Intelligent use of the Factor Theorem (§ 114) is often useful.

Consider $x^3-3x^2-10x+24$.

If there is a factor of the form $x-a$, a must be a factor of 24; therefore possible values for a are 1, 2, 3, 4, 6, 12, 24, or -1, -2, -3, -4, -6, -12, -24.

Denoting the original expression by $f(x)$, we have

$f(1) = 1-3-10+24 \neq 0$; $\therefore$ $x-1$ is not a factor.

$f(-1) = -1-3+10+24 \neq 0$; $\therefore$ $x+1$ „

$f(2) = 8-12-20+24 = 0$; $\therefore$ $x-2$ is a factor.

The other factor may be obtained by division.

$$\begin{array}{c|l} 1 & 1-3-10+24 \\ 2 & \quad +2-\ 2-24 \\ \hline & x^2-x-12 \qquad \text{The other factor.} \end{array}$$

This last is Case IV.

Hence $f(x) = (x-2)(x-4)(x+3)$.

Instead of division the following process may be used when we are certain of one factor.

$$\begin{aligned} f(x) &= x^3(x-2) - x(x-2) - 12(x-2). \\ &= (x-2)(x^2-x-12) \\ &= (x-2)(x-4)(x+3). \end{aligned}$$

Here we begin by writing $x^3(x-2)$. This has taken into account x^3, but accounts for $-2x^2$ only; hence there is left $-x^2$. Therefore there must follow $-x(x-2)$, for this makes the x^2 term correct. But we have now brought in a term $+2x$ so; in order to make $-10x$, $12x$ must be subtracted. Hence we finish with $-12(x-2)$; this makes the correct coefficient for x, and at the same time brings in the right constant, viz. $+24$. With a little practice this method can be used very quickly, and it looks neater than the division sum.

Instead of dividing by the factor $x-2$, the other factors could be found by continued use of the factor theorem. The objection to this is that sometimes the remaining quadratic factor cannot be resolved into factors of the form $x-a$, and so the time given to trying the factor theorem would be wasted, e. g.

$$f(x) \equiv x^3+5x^2+10x+8.$$

a may be ± 1, ± 2, ± 4, ± 8.

$f(1) = 1+5+10+8 \neq 0$, $\therefore$ $x-1$ is not a factor;

$f(-1) = -1+5-10+8 \neq 0$, $\therefore$ $x+1$ is not a factor;

$f(-2) = -8+20-20+8 = 0$, $\therefore$ $x+2$ is a factor;

$$\begin{aligned} \therefore f(x) &= x^2(x+2)+3x(x+2)+4(x+2) \\ &= (x+2)(x^2+3x+4); \end{aligned}$$

and it is found, by using the second method for Case IV, that x^2+3x+4 has no factors for

$$\begin{aligned} x^2+3x+4 &= x^2+3x+(\tfrac{3}{2})^2-(\tfrac{3}{2})^2+4 \\ &= (x+\tfrac{3}{2})^2+\tfrac{7}{4}. \end{aligned}$$

This is the sum of two squares, and therefore has no real factors.

129. It is understood when finding factors that the coefficients and constants in the factors are to be real and rational. The meaning of this can be seen from the following examples, where it is supposed that there is no restriction.

$$\begin{aligned} x^2-7 &= x^2-(\sqrt{7})^2 \\ &= (x+\sqrt{7})(x-\sqrt{7}). \end{aligned} \qquad \text{Case III.}$$

Here $\sqrt{7}$ can be found to any required number of places by ordinary arithmetic, so that the factors are quite *real*, but $\sqrt{7}$

cannot be found exactly, and is therefore not *rational*. Hence, because 7 is not an exact square, x^2-7 has no rational factors .

$$\begin{aligned} x^2+7 &= x^2-(-7) \\ &= x^2-(\sqrt{-7})^2 \\ &= (x+\sqrt{-7})(x-\sqrt{-7}). \end{aligned}$$

But all real numbers, whether positive or negative, are positive when squared, therefore $\sqrt{-7}$ is not a real number. Hence x^2+7 has no real factors.

Examples XIV f.

Resolve into as many factors as possible:

1. $4x^4-9a^4$.
2. $25x^4-16y^4$.
3. $16a^4-81b^4$.
4. x^4+y^4.
5. $1-625a^4$.
6. $16-49k^4$.
7. x^4-5x^2-6.
8. $x^4+2x^2y^2+9y^4$.
9. $x^4-15x^2y^2+9y^4$.
10. $16x^4+36x^2y^2+81y^4$.
11. $a^4-a^2b^2+b^4$.
12. ax^3-bx^2+bx-a.
13. $20x^3-28x^2-15x+21$.
14. $12a^3+16a^2-3ax-4x$.
15. $x^3-6x^2+11x-6$.
16. $x^4-3x^3+4x^2-6x+4$.
17. x^3-x^2-x+1.
18. x^3-3x+2.
19. $15+19x-4x^2$.
20. x^3-3x^2-x+3.
21. x^4-10x^2+9.
22. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
23. $x^3+3x^2-4x-12$.
24. $x^3+x^2+x-y^3-y^2-y$.
25. $(a-b)^3+(b-c)^3$.
26. $x^3+5x^2-4x-20$.

What values of x will make the following expressions respectively equal to nothing?

27. x^2-256.
28. $x^2-13x-30$.
29. $(x-3)(x^2-x-30)$.
30. x^3-4x^2+x+6.
31. x^3-343.
32. x^4-81.

33. If $x+2$ is a factor of $x^4-2x^3-30x^2-47x-6$, what are the remaining factors?

34. Being given that $x+6$ is one factor, find the other factors of

$$x^4+5x^3-20x^2-60x+144.$$

Examples XIV.

Find the factors of:

1. $(x-2)(2x^2-7x+6)+(x-2)(2x-3)$.
2. x^2+x-20.
3. $6x^2-7x+2$.
4. $12x^2-17x+6$.
5. $x^3-2x^2y+2xy^2-y^3$.
6. $3x^2+5x-2$.

7. $x^2-(y-2z)^2$.
8. $(a+2b)^3+(a-2b)^3$.
9. $12x^2y^2-17xy+6$.
10. $x^4+2x^3+2x^2+2x+1$.
11. $4x^2-5x+1$.
12. $ax-ab-bx+b^2$.
13. $x^4+x^3-12x^2$.
14. $4x^2-9$.
15. $x^3-2x^2y-4xy^2+8y^3$.
16. $16x^4-20x^2+4$.
17. $a^2-b^2+2ac+c^2$.
18. x^3-16xy^2.
19. $ad+bx+ax+bd$.
20. $6x^2-ax-2a^2$.
21. $ac+2by+2ay+bc$.
22. $3x^3-8x^2-x+6$.
23. $2x^3-9x^2+4x+12$.
24. $15(x-y)(x+y)-16xy$. 16xy.
25. $(a^4-b^4)\{c^3-(d-1)^3\}$.
26. $x^4-10x^2y^2+9y^4$.
27. $a-bc-b(1-c)$.
28. $10x^2+13x-6$.
29. $15x^2-13x+2$.
30. $(a+2b-3c)^2-(3a-b+2c)^2$.
31. x^2-x-42.
32. $9a^2+6ab-16c^2-8bc$.
33. $(3a+2b+c)^2-(a+2b+3c)^2$.
34. $3x^3-x^2-6x+2$.
35. x^4+x^2+1.
36. $bc(b-c)+ca(c-a)+ab(a-b)$.
37. $x^2-(y+1)^2$.
38. $(x^2-y^2)^2-2(x^2+y^2)+1$.
39. $x^2-22x-15$.
40. $(a+c)(a-c)-b(2a-b)$.

Find in as few steps as possible the following products:

41. $(x-1)(x+1)(x^2+1)(x^4+1)$.
42. $(x-1)(x+1)(x^2+x+1)(x^2-x+1)$.
43. $(x-2)(x-3)(x-1)(x-6)$.
44. $(2x+1)(4x^2+2x+1)(4x^2-2x+1)(2x-1)$.
45. $(x-3)(x-4)(x-1)(x-6)$.
46. $(x-a)(x-b)(x^2+ab)$.
47. $(x-2a)(2x-a)(2x+2a)(x+a)$.

Find an expression that will exactly divide both

48. x^3-3x+2 and $x^3-8x+12$.
49. x^3-3x+2 and x^3-x-6.
50. $10x^2-x-21$ and $14x^3-11x-15$.
51. $3x^2+14x-5$ and $3x^3+8x^2-1$.
52. $a^2-2ab+b^2+a-b$ and $a^2-2ab+b^2+a^2-b^2$.

Find values of x that will make the following expressions respectively equal to nothing.

53. $5(x-3)-4(x-3)$.
54. x^2-8x-9.
55. $x^2-13x-48$.
56. $3x^2+14x-5$.
57. $6x^2+11x-10$.
58. $x^3-2x^2-21x-18$.
59. $x^3-x^2-30x+72$.

Prove that the following expressions have no real factors:

60. x^2+2x+7.
61. x^2-4x+9.
62. $3x^2-6x+5$.
63. $4x^2+3x+2$.

Resolve the following expressions into factors, which may in some cases be irrational:

64. x^2-2x-7. 65. x^2-3x-5.

66. $3x^2-8x+5$. 67. $4x^2-9x+4$.

68. $5x^2+2x-7$. 69. $7x^2-12x+4$.

Resolve into factors:

70. $\frac{1}{a^2}-\frac{7}{ab}-\frac{18}{b^2}$. 71. $\frac{x^3}{y^3}-\frac{z^3}{8}$.

72. $6(3x-y)^2-5(3x-y)(a-2b)+(a-2b)^2$.

73. $x^8+64y^8+16x^3y^3-1$. 74. $8x^3+\frac{343}{x^6}$.

75. $(2a-5)^4+(2a-5)^2(3a+4)^2+(3a+4)^4$.

CHAPTER XV

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

130. A common factor of two or more expressions is a factor which divides each of them. Thus a and b and ab are, each of them, common factors of the two expressions abc and bcd; $(x+a)$ and (x^2+a^2) are common factors of $(x+a)(x+b)(x^3+a^2)$ and $(x+a)(x+c)(x^2+a^2)$.

The common factor which is of highest dimensions is called the **highest common factor**, and is usually denoted by the capital letters H. C. F.

A common multiple of two or more expressions is a quantity which can be exactly divided by each of those expressions. Thus a^2b^2c is a common multiple of abc and ab^2; x^4-a^4 is a common multiple of $(x+a)$ and $(x-a)$.

The common multiple which is of lowest dimensions is called the lowest common multiple, and is usually denoted by the capital letters L. C. M. The L. C. M. of abc and ab^2 is therefore ab^2c, and not a^2b^2c; the L.C.M. of $(x+a)$ and $(x-a)$ is x^2-a^2.

Numbers have no dimensions, so when dealing with numbers we use the adjectives 'greatest' and 'least' instead of 'highest' and 'lowest', and talk about 'greatest common measure' (G. C. M.), 'least common multiple' (L. C. M.), instead of 'highest common factor', 'lowest common multiple.'

In all respects H. C. F. and L. C. M. serve the same purposes in Algebra as G. C. M. and L. C. M. do in Arithmetic; and the methods for obtaining them are the same in principle as the Arithmetical methods for finding G. C. M. and L. C. M.

Examples XV a.

1. Resolve 30 and 140 into factors; write down all the common factors and the G. C. M.

2. Write down all the common factors of a^2b^3x and abx^2. Hence find the H. C. F.

3. Find the H. C. F. of $30a^2b^3x$ and $140abx^2$.

4. What expressions are common factors of $x(x^2-7x+12)$ and $x^2(x^2+7x-30)$? Find the H. C. F.

5. Find the H. C. F. of $15x(x^2-7x+12)$ and $27x^2(x^2+7x-30)$.

6. Find three common multiples of 16 and 24. What is the L. C. M.?

7. Write down three common multiples of a^2bc^2 and ab^4c^3. Find their L. C. M.

8. Find the L. C. M. of $16a^2bc^2$ and $24ab^4c^3$.

9. Write down three common multiples of x^2-4 and x^2-5x+6. Find the L. C. M.

10. Find the L. C. M. of $15(x^2-4)$ and $21(x^2-5x+6)$.

131. A slight difficulty may have been noticed in some of the preceding examples. Suppose it were required to find the H. C. F. of $16xy^2z$ and $24x^2yz^2$; the H. C. F. of xy^2z and x^2yz^2 is xyz so that $4xyz$, $2xyz$, $8xyz$ are all common factors of $16xy^2z$ and $24x^2yz^2$, and they are of the same dimensions. In such cases it is understood that the coefficient of the required H. C. F. is the G. C. M. of the coefficients of the various expressions; so in this particular example the H. C. F. is $8xyz$.

Similarly, the coefficient of the L. C. M is the L. C. M. of the coefficients of the various expressions.

132. The methods of finding the H. C. F. and L. C. M. can be seen from the following examples.

Example I. *Find the* H. C. F. *of* $8a^2x^3$, $9ax^4y^2$, $6a^2x^2y^2$.

The G. C. M. of the coefficients is 3.

The only letters common to all three are a and x.

$\therefore$ the H. C. F. must contain some powers of a and x.

One of the expressions, viz. the second, contains only the first power of a, $\therefore$ the H. C. F. can contain only the first power of a.

The lowest power of x in the three expressions is x^2, i. e. the highest power of x that is common to all of them is x^2, i. e. the H. C. F. is $3ax^2$.

Similar examples could be written out at length in the same way, but the whole work is usually done mentally.

Example II. *Find the* L. C. M. *of* $8a^2x^3$, $9ax^4y^2$, $6a^2x^2y^2$.

The L. C. M. of the coefficients is 36.

In order that the L. C. M. may be divided by a^2x^3 it cannot contain lower powers than a^2x^3.

In order that the L. C. M. may be divided by ax^4y^2 there must be y^2 and another x in addition to a^2x^3, i. e. there must be $a^2x^4y^2$, and we see that $a^2x^4y^2$ can be divided by $a^2x^2y^2$.

Hence the L. C. M. is $36a^2x^4y^2$.

The work is usually done mentally.

Example III. *Find the* H. C. F. *and* L. C. M. *of*

$$x^2-x-56 \text{ and } x^2+10x+21.$$

$$x^2-x-56=(x-8)(x+7),$$
$$x^2+10x+21=(x+3)(x+7);$$
$$\therefore \text{ H. C. F.}=(x+7),$$
$$\text{and L. C. M.}=(x+3)(x+7)(x-8).$$

Example IV. *Find the* H. C. F. *and* L. C. M. *of*

$$x^2-4x+4,\ x^2-10x+16,\ x^2+x-6.$$

$$x^2-4x+4=(x-2)^2,$$
$$x^2-10x+16=(x-2)(x-8),$$
$$x^2+x-6=(x-2)(x+3);$$
$$\text{H. C. F.}=(x-2),$$
$$\text{L. C. M.}=(x-2)^2(x+3)(x-8).$$

Example V. *Find the* H. C. F. *and* L. C. M. *of*

$$2x^3-11x^2+20x-12 \text{ and } 3x^3-7x^2-8x+20.$$

Both expressions must be factorized; and the Factor Theorem should be used.

By comparing the two coefficients of x^3, it is seen that if there is a common factor the first coefficient must be unity.

By comparing the two constant terms it is seen that the constant of any common factor must be ± 1, ± 2, ± 4. This is all done mentally so far; the work should be set down as follows:

For first expression $f(1) = 22 - 23 \neq 0$,

$$f(-1) = -2 - 11 - 20 - 12 \neq 0,$$

$$f(2) = 16 - 44 + 40 - 12 = 0.$$

$\therefore$ $x-2$ is a factor.

$$\begin{aligned} 2x^3 - 11x^2 + 20x - 12 &= 2x^2(x-2) - 7x(x-2) + 6(x-2) \\ &= (x-2)(2x^2 - 7x + 6) \\ &= (x-2)(2x-3)(x-2) \\ &= (x-2)^2(2x-3). \end{aligned}$$

Hence the only possible common factor is $x-2$ or $(x-2)^2$; try $x-2$.

$$\begin{aligned} 3x^3 - 7x^2 - 8x + 20 &= 3x^2(x-2) - x(x-2) - 10(x-2) \\ &= (x-2)(3x^2 - x - 10) \\ &= (x-2)^2(3x+5). \end{aligned}$$

Hence H. C. F. $= (x-2)^2$,

L. C. M. $= (x-2)^2(2x-3)(3x+5)$.

Examples XV.

Find the H. C. F. and L. C. M. of:

1. $3a^2bc^2$, $6ab^3c^4$, $8a^3bc^3$, $2a^3c^3$.
2. $21x^3y^4z^5$, $12x^4y^3z^3$, $15x^5y^3z^3$.
3. $50a^2x^2y^2$, $20a^3xy^3$, $25a^2x^3y^4$.
4. $5x^3(x-2)$, $10x^2(x-3)$, $16x^5(x^2-5x+6)$.
5. $3x^2(x+1)$, $21x^4(x^2+2x+1)$, $9x^3(x^2-1)$.
6. $a^4(x^2-1)$, $3a^2(x^2-1)$, $6a^5(x^4-1)$.
7. $3a(x^2+2x-8)$, $a^2(x^2-x-20)$, $6a^3(x^2-3x-28)$.
8. $4a^2bc$, $5ac^2x$, $6cx^2y$, $7x^2y^2z^2$.
9. x^2-3x+2, $x^2+10x-11$, x^2-1.
10. $x^2-25x+144$, $x^2-9x-112$.

Find the H. C. F. of:

11. $3x^2+x-2$ and $3x^2+4x-4$.
12. $2x^2+7x-15$ and $x^2+9x+20$.
13. x^3-2x-3 and x^3-x^2-7x+3.

14. x^2-3x+2 and x^3-5x+2.
15. x^2-x-2 and $x^4+x^3-5x^2+x-6$.
16. x^3-2x^2+x-2 and x^4+4x^2+3.
17. x^3+x^2-x-1 and x^3-3x-2.
18. $6x^3+19x-7$ and $10x^3+21x^2-47x+7$.
19. x^3-4x^2+5x-2 and x^3-x^2-x-2.
20. $3x^3+4x^2-x-2$ and x^3-x^2-5x-3.

Find the L. C. M. of:

21. $4a(a+b)$ and $6ab(a^2-ab-2b^2)$.
22. a^3-a^2b, a^2b+ab^2, a^2b-b^3.
23. $x^2+ax-2a^2$, $x^2-2ax+a^2$, $2x^2-ax-a^2$.
24. x^2-1 and x^5+1.
25. $2x^2+7ax-15a^2$, $2x^2+11ax-21a^2$, and $5x^3+59ax^2+163a^2x-35a^3$.
26. $2x^2-x-6$, $3x^2-7x+2$, $6x^2+7x-3$.
27. x^2-1, x^2-6x-7, x^3-3x^2+2x, $3x^2-7x+2$.
28. x^4-x^2, x^4-x, x^4+x^2+1.
29. $2x^3+x-3$, x^3+2x-3, x^3+3x^2-x-3.
30. $6y^2-y-2$, $3y^2+5y+2$, $3y^3-y^2+y+2$.

CHAPTER XVI

FRACTIONS

133. Definition. When a quantity p is divided by another quantity q, the result may be written $\frac{p}{q}$. When the division cannot be performed exactly, this form is called a **fraction.** The quantities p and q may be either numbers or algebraical expressions; thus

$$\frac{3}{5}, \quad \frac{2x}{3}, \quad \frac{x+a}{3x-a}, \quad \frac{x^2-ax+a^2}{x^2+ax+a^2}$$

are all fractions.

In Arithmetic $\frac{3}{5}$ is regarded as being three times the result of dividing unity by five, and is called three-fifths; hence the lower number gives the name to the parts, in this case 'fifths', and is called the denominator; the upper number shows the number of

parts to be taken, and is called the numerator. These names, 'numerator' and 'denominator', are used in Algebra also to denote the dividend and divisor respectively.

An integral expression is an expression containing no fractions with algebraical denominators.

Note. The result obtained by dividing a number p by a number q is clearly p times as great as the number obtained by dividing unity by q. Hence follow the two ways of looking at numerical fractions; $\frac{3}{5}$ may be regarded as the result obtained by dividing 3 by 5 or as three times the result obtained by dividing 1 by 5.

134. The value of a fraction is unaltered by multiplying or dividing both numerator and denominator by the same quantity.

This follows from the axiom that if equals be multiplied by equals, the products are equals.

For suppose $qx = p$, then $x = \frac{p}{q}$.

Multiply both sides by m,

then $mqx = mp$ and $x = \frac{mp}{mq}$.

Hence $$\frac{p}{q} = \frac{mp}{mq}.$$

Also since $\frac{mp}{mq} = \frac{p}{q}$, **it is seen that both numerator and denominator may be divided by the same quantity without altering the value of the fraction (provided that quantity is not equal to nothing, § 108).**

It follows from this rule that factors common to both numerator and denominator may be divided out just as in Arithmetic; and that any number of fractions may be reduced to a common denominator.

Examples XVI a.

Reduce to their simplest form:

1. $\frac{3abx^2y}{15a^2bx}$.
2. $\frac{5a^2b^3x^4}{2ab^3x^3}$.
3. $\frac{24a^4b^5c^7}{18a^5b^3c^5}$.

4. $\frac{x^2-a^2}{x^2-2ax+a^2}$. 5. $\frac{a^2-2ab-3b^2}{a^2-4ab-5b^2}$. 6. $\frac{x^2-2ax-48a^2}{x^2+ax-42a^2}$.

7. $\frac{x^3+2x^2-x-2}{x^3+3x^2-4x-12}$. 8. $\frac{xy+4ab+2ay+2bx}{xy-4ab+2ay-2bx}$.

9. $\frac{a^2-2ab+b^2+a-b}{a^2-2ab+b^2+a^2-b^2}$.

Express the following fractions as fractions with the same denominator:

10. $\frac{2}{a}, \frac{3}{b}, \frac{x}{ab}$. 11. $\frac{x^2}{3ab}, \frac{2y^2}{5bc}, \frac{3xy}{4ac}$.

12. $\frac{x-a}{x+a}$ and $\frac{x+a}{x-a}$. 13. $\frac{a-2b}{a-b}$ and $\frac{a+2b}{a+b}$.

14. $\frac{2x}{3a}$ and $\frac{2x-3a}{2x+3a}$. 15. $\frac{p}{p+2q}$ and $\frac{q}{p-2q}$.

16. By taking a rectangle 5 by 1 to represent unity, show that three divided by five equals three-fifths.

17. By taking a rectangle 5 by 4 to represent unity, show graphically that $\frac{1}{5} = \frac{4}{20}$.

18. Show graphically that $\frac{2}{4} = \frac{6}{12}$ and that $\frac{8}{12} = \frac{2}{3}$.

19. Show graphically that $\frac{2}{3} + \frac{1}{3} = \frac{11}{12}$.

By reducing to the least common denominator, simplify:

20. $\frac{1}{2a} + \frac{2}{3a} - \frac{3}{5a}$. 21. $\frac{a}{2b} + \frac{2b}{3a}$.

22. $\frac{3a}{4b} + \frac{2b}{3c} - \frac{5c}{12a}$. 23. $\frac{3}{ab} + \frac{2}{bc} - \frac{3}{ac}$.

24. $\frac{3x}{ab} + \frac{2y}{bc} - \frac{3z}{ac}$. 25. $\frac{3ab}{x} + \frac{2bc}{y} - \frac{3ac}{z}$.

26. $\frac{ab}{3x} + \frac{bc}{2y} - \frac{ac}{3z}$. 27. $\frac{2ab}{3x} - \frac{5bc}{2y} + \frac{4ac}{3z}$.

28. $\frac{1}{x+a} + \frac{1}{x-a}$. 29. $\frac{x}{x+a} - \frac{x}{x-a}$.

Just as $\frac{19}{5}$ can be reduced to the form $3\frac{4}{5}$, reduce the following fractions to similar forms:

30. $\frac{2a+3}{a}$. 31. $\frac{3x+4a}{x}$. 32. $\frac{3x^2-4ax+5a^2}{x}$.

33. $\frac{2a+3b}{a+b}$. 34. $\frac{3a-4c}{a-c}$. 35. $\frac{4a-7x}{a+x}$.

36. $\frac{a^2-2ab+3b^2}{a-b}$.

37. $\frac{a^2+3ab+3b^2}{a+2b}$.

38. $\frac{a^3-5ab+6b^3}{(a-6b)}$.

Solve:

39. $\frac{x}{a}-\frac{1}{b}=\frac{x}{b}-\frac{1}{a}$.

40. $\frac{x}{ab}+\frac{3}{bc}=\frac{1}{ac}-\frac{x}{ab}$.

41. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$.

42. $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{b}$, $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{b}$.

135. Addition and Subtraction of Fractions.

Suppose $x=\frac{p}{q}$ and $y=\frac{r}{s}$,

then $x=\frac{ps}{qs}$ and $y=\frac{qr}{qs}$.

$\therefore$ $qsx=ps$,

and $qsy=qr$.

Add $qs(x+y)=ps+qr$.

$\therefore$ $x+y=\frac{ps+qr}{qs}$.

The same method can be applied however many fractions there may be, and whether the connecting signs are + or −. Hence follows the rule:

To add or subtract fractions. Reduce all the fractions to the common denominator of lowest dimensions, and then add or subtract the new numerators.

As a rule the common denominator is left in factors and not multiplied out.

As is pointed out in § 78 the line of a fraction has the effect of a vinculum or bracket; so that a minus sign before the fraction changes the sign of all the terms in the numerator. If examples are at first worked as in the following example, there is less chance of making a mistake.

$$\frac{x-a}{b}-\frac{x-b}{a}=\frac{a(x-a)-b(x-b)}{ab}$$
$$=\frac{ax-a^2-bx+b^2}{ab}$$
$$=\frac{(a-b)x-(a^2-b^2)}{ab}.$$

When the student has had a fair amount of practice the first step may be omitted.

$$\frac{2x+3a}{2b}-\frac{3x-2b}{2a}=\frac{2ax+3a^2-3bx+2b^2}{2ab}$$
$$=\frac{(2a-3b)x+(3a^2-2b^2)}{2ab}.$$

Simplify $-\dfrac{x-10}{x^2+x-12}-\dfrac{x+22}{x^2-x-20}+\dfrac{x+1}{x^2-8x+15}$.

Expression:

$$=-\frac{x-10}{(x+4)(x-3)}-\frac{x+22}{(x-5)(x+4)}+\frac{x+1}{(x-3)(x-5)}$$
$$=\frac{-(x-10)(x-5)-(x+22)(x-3)+(x+1)(x+4)}{(x-3)(x+4)(x-5)}$$
$$=\frac{-(x^2-15x+50)-(x^2+19x-66)+(x^2+5x+4)}{(x-3)(x+4)(x-5)}$$
$$=\frac{-x^2+15x-50-x^2-19x+66+x^2+5x+4}{(x-3)(x+4)(x-5)}$$
$$=\frac{-x^2+x+20}{(x-3)(x+4)(x-5)}$$
$$=-\frac{x^2-x-20}{(x-3)(x+4)(x-5)}$$
$$=-\frac{(x-5)(x+4)}{(x-3)(x+4)(x-5)}$$
$$=-\frac{1}{x-3}.$$

The signs of equality should be kept in a straight line one under the other. After a time the student will be able to leave out some of the steps, but he should always recollect that accuracy is more important than speed, and that untidy, badly-arranged work is more likely to be wrong than neat work.

Examples XVI b.

Simplify the following expressions:

1. $\frac{a-b}{ab}+\frac{c-a}{ac}+\frac{b-c}{bc}$.

2. $\frac{b^2c^2}{d}+\frac{a^2c^2}{b}-\frac{b^2d^2}{c}$.

3. $-\frac{x+1}{2}-\frac{3x+1}{4}+\frac{5x+1}{6}$.

4. $\frac{2x+3}{5}-\frac{3x-2}{4}+\frac{7x-2}{20}$.

5. $\frac{2x-3}{3}-\frac{3x-5}{5}+\frac{5x+3}{6}-\frac{7x+5}{10}$.

6. $\frac{x(1-3x)}{3}-\frac{3(x-2)}{4}-\frac{x^2+3}{6}-\frac{7(x^2-1)}{36}$.

7. $\frac{1-x}{2}-\frac{2(x-2)}{3}+\frac{3(1-2x)}{4}$.

8. $\frac{1}{x+1}+\frac{1}{x+5}$.

9. $\frac{3}{x-1}-\frac{1}{x+3}$.

10. $\frac{2a^2}{a^2-b^2}+\frac{b}{a+b}$.

11. $\frac{2a-b}{a-b}-\frac{2a+b}{a+b}$

12. $\frac{3}{x+2}-\frac{2}{x+3}+\frac{x+1}{(x+2)(x+3)}$.

13. $\frac{x-2}{x-1}-\frac{x-1}{x-2}-\frac{3}{(x-1)(x-2)}$.

14. $\frac{x-1}{2x}-\frac{x^2-2x+1}{4x^2}-\frac{x^2-x-1}{4x^2}$.

15. $\frac{3}{x-3}-\frac{1}{x+5}-\frac{2}{x-7}$.

16. $\frac{y}{x(y-x)}-\frac{x}{y(x+y)}+\frac{2y}{x^2-y^2}$.

17. $\frac{x}{x^2-3x+2}+\frac{2}{x^2-1}-\frac{6}{x^2-x-2}$.

18. $\frac{3}{1+a}-\frac{2}{1-a}-\frac{5a}{a^2-1}$.

19. $\frac{3}{x+1}-\frac{2}{x^2+3x+2}-\frac{2}{x+2}$.

20. $\frac{6a}{a^2-4}-\frac{2}{2+a}+\frac{3}{2-a}$.

136. Multiplication and Division of Fractions.

Suppose $x = \frac{p}{q}$ and $y = \frac{r}{s}$,

then $qx = p,$

$sy = r.$

Multiply together the two left-hand sides and the two right-hand sides,

$$qsxy = pr.$$

Divide by qs, $$xy = \frac{pr}{qs}.$$

Any number of fractions could be dealt with in a similar way.

Hence the product of any number of fractions is a fraction whose numerator is the product of the original numerators and whose denominator is the product of the original denominators. By § 184, any factor common to the numerator and denominator may be divided out.

Again if $x = \frac{p}{q}$ and $y = \frac{r}{s}$,

then $qx = p$

and $sy = r.$

Dividing $\frac{qx}{sy} = \frac{p}{r}.$

Multiply by s $\frac{qx}{y} = \frac{ps}{r}$

Divide by q $\frac{x}{y} = \frac{ps}{qr} = \frac{p}{q} \times \frac{s}{r}.$

Hence we have the rule for division:

To divide by any fraction invert the fraction and multiply.

In all cases involving multiplication and division of fractions, the first step is to resolve all the numerators and denominators into factors, and then cancel factors common to the numerator and denominator.

Example I.

Simplify $\dfrac{a^2+ab}{a^2-ab} \times \dfrac{ab-b^2}{ab+b^2} \times \dfrac{ab-bc}{ab+bc}$.

$$\text{Expression} = \frac{a^2+ab}{a^2-ab} \times \frac{ab-b^2}{ab+b^2} \times \frac{ab-bc}{ab+bc}$$

$$= \frac{\cancel{a}(a+b)}{\cancel{a}(a-b)} \times \frac{\cancel{b}(a-b)}{\cancel{b}(a+b)} \times \frac{\cancel{b}(a-c)}{\cancel{b}(a+c)}$$

$$= \frac{a-c}{a+c}.$$

Examples XVI c.

Simplify the following expressions :

1. $\dfrac{2ab^2c}{3a^2bc^2} \times \dfrac{4a^2bc^3}{5ab^2c^4} \times \dfrac{3abc}{2a^2bc^3}$.
2. $\dfrac{3ab}{2cx} \times \dfrac{5by}{4ax} \times \dfrac{3cz}{2bx}$.
3. $\dfrac{a(x-b)}{bx+ab} \times \dfrac{b^2x+ab^2}{a^2x+a^3} \times \dfrac{x^2-a^2}{x^2-b^2}$.
4. $\dfrac{3ab^2}{4bc^2} \times \dfrac{2b^2c}{7ac} \div \dfrac{3a^2b^2}{4bc}$.
5. $\dfrac{3ab}{2cx} \times \dfrac{5by}{4ax} \div \dfrac{3cz}{2bx}$.
6. $\dfrac{x^2-5x+6}{x^2-16} \times \dfrac{x^2+7x+12}{x^2-8x+15} \times \dfrac{x^2+x-20}{x^2+x-6}$.
7. $\dfrac{4x^2-9}{2x^2+5x+3} \div \dfrac{2x-3}{x(x+1)}$.
8. $\dfrac{6x^2-ax-2a^2}{ax-a^2} \times \dfrac{x-a}{9x^2-4a^2} \div \dfrac{a+2x}{a(2a+3x)}$.
9. $\dfrac{x^2-(y-z)^2}{y^2-(z-x)^2} \div \dfrac{(y+z)^2-x^2}{z^2-(x-y)^2}$.
10. $\dfrac{a^4-b^4}{a^2-2ab+b^2} \times \dfrac{a-b}{a^2+ab}$.
11. $\dfrac{2x^2+x-1}{x^2-4x+3} \times \dfrac{2x^2-5x+3}{6x^2+x-2} \times \dfrac{3x^2-7x-6}{2x^2-7x+6}$.
12. $\dfrac{(x-y)^2-z^2}{xy-y^2-yz} \times \dfrac{z}{x^2+xy-xz} \div \dfrac{xz-yz+z^2}{x^2-(y-z)^2}$.

13. $\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\div\left(\frac{1}{a}-\frac{1}{b}\right)$.

14. $\left(\frac{1}{2a}+\frac{1}{3b}\right)\times 6ab$.

15. $\frac{x+2a}{2x-3a}\left(\frac{1}{x+2a}+\frac{1}{x-5a}\right)$.

16. $\left(\frac{1}{a}+\frac{1}{ab^2}\right)\div\left(b-1+\frac{1}{b}\right)$.

17. $\left(\frac{a}{a-x}-\frac{x}{a+x}+\frac{2ax}{a^2-x^2}\right)\div(a+x)^2$.

18. What must $\frac{x-3}{x-4}$ be multiplied by to give $\frac{x^2-5x+6}{x^2-7x+12}$ as product?

19. Find the quotient when $\frac{x+4}{x-1}$ is divided by $\frac{3x^2+10x-8}{4x^2+x-5}$.

20. Solve $\frac{(a-b)x}{a+b}=a^2-b^2$.

21. Find the missing term in the following identity:

$$\frac{x^2-5x+6}{x^2-3x-4}\times\frac{x^2-x+\quad}{x^2+2x-15}\div\frac{x^2+5x-14}{x^2+6x+5}=\frac{x^2-9}{x^2+4x-21}.$$

22. What must $\frac{x-2}{x-3}-\frac{x-4}{x-5}$ be multiplied by in order that the product shall contain no fractions? What is the simplest form of the product?

23. Solve the equations:

(i) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=\frac{x}{(x-3)(x-5)}$;

(ii) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=\frac{x^2-8x+13}{(x-3)(x-5)}$;

(iii) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=-\frac{2}{(x-3)(x-5)}$;

(iv) $\frac{x-2}{x-3}-\frac{x-4}{x-5}=0$.

24. Simplify $\left(1+\frac{b}{a}\right)\left(1+\frac{c}{a}\right)\div\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right)$.

25. Simplify $\left(1+\frac{2a}{1+a}\right)\left(1+\frac{2a}{1-a}\right)$.

137. Special points in connexion with fractions.

(i) Reduce all fractions to their lowest terms before adding or subtracting.

(ii) Factorize all numerators or denominators before multiplying or dividing.

(iii) $\frac{a}{b} \times b = a$, $\frac{a}{b} \times bc = ac$. That is:

Any fraction can be reduced to an integral expression by multiplying it by its denominator or by a multiple of its denominator.

Hence the algebraic sum of any number of fractions can be reduced to an integral form by multiplying by the L. C. M. of all the denominators.

This rule is continually being used when equations that contain fractions are to be solved.

(iv) The value of a fraction is unaltered when the numerator and denominator are both multiplied by the same quantity.

This has already been given as the first rule in dealing with fractions. It is here mentioned again as number (iv) because, together with number (iii) above, it gives the easiest method of simplifying complicated fractions.

138. Miscellaneous worked examples in Fractions.

Example I.

Simplify $\dfrac{x^2-2ax+a^2}{x^2-a^2} + \dfrac{3ax-15a^2}{x^2-4a-5a^2} - \dfrac{3x^2+4ax-4a^2}{x^2+3ax+2a^2}$.

Expression

$$= \frac{(x-a)^2}{(x-a)(x+a)} + \frac{3a(x-5a)}{(x+a)(x-5a)} - \frac{(3x-2a)(x+2a)}{(x+a)(x+2a)}$$

$$= \frac{x-a}{x+a} + \frac{3a}{x+a} - \frac{3x-2a}{x+a}$$

$$= \frac{-2x+4a}{x+a}$$

$$= -\frac{2(x-2a)}{x+a}.$$

Example II. *Simplify* $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$.

$$\text{Fraction} = \frac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$$

$$= \frac{a^2-b^2}{a^2+b^2}.$$

(Multiplying num. and den. by ab, § 137 iv, in order to clear them of fractions, § 137 iii.)

Example III. *Simplify* $\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{x}{x^2-a^2}$.

[In all examples it is advisable to keep the letters in the same order; here a comes before x in two fractions but x before a in the third fraction. The first step is to get the same order throughout. Recollect that $a+b=b+a$, but $a-b=-(b-a)$.]

$$\text{Expression}=\frac{1}{a+x}+\frac{1}{a-x}+\frac{-x}{a^2-x^2}$$ (Multiplying num. and den. of third fraction by -1.)

$$=\frac{a-x+a+x-x}{a^2-x^2}$$

$$=\frac{2a-x}{a^2-x^2}.$$

If we multiply both numerator and denominator by -1, we see that the answer may be written $\dfrac{x-2a}{x^2-a^2}$.

Example IV.

$$\textit{Simplify}\ \left(1-\frac{2ab}{a^2+b^2}\right)\times\frac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}\div\left(a-\frac{2ab}{a+b}\right).$$

[The quantities in brackets should be brought to a common denominator before the brackets are removed. Such expressions as this should be kept as single expressions throughout the work, and the parts should not be simplified by separate sums.]

Wrong way.

$$1-\frac{2ab}{a^2+b^2}=\frac{a^2+b^2-2ab}{a^2+b^2},\qquad a-\frac{2ab}{a+b}=\frac{a^2+ab-2ab}{a+b}=\frac{a^2-ab}{a+b},$$

$$\frac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}=\frac{a^2+b^2}{a^2-b^2}.$$

$$\text{Expression}=\frac{a^2-2ab+b^2}{a^2+b^2}\times\frac{a^2+b^2}{a^2-b^2}\times\frac{a+b}{a^2-ab}.$$

&c.

Right way.

$$\text{Expression} = \left(1 - \frac{2ab}{a^2+b^2}\right) \times \frac{\frac{a}{b}+\frac{b}{a}}{\frac{a}{b}-\frac{b}{a}} \div \left(a - \frac{2ab}{a+b}\right)$$

$$= \frac{a^2+b^2-2ab}{a^2+b^2} \times \frac{a^2+b^2}{a^2-b^2} \div \frac{a^2+ab-2ab}{a+b}$$

$$= \frac{(a-b)^2}{(a-b)(a+b)} \times \frac{a+b}{a(a-b)}$$

$$= \frac{1}{a}.$$

In the first line each expression in brackets is reduced to its common denominator, and the second fraction is simplified by multiplying throughout by ab.

In the second line the common factor a^2+b^2 is omitted, and the numerators and denominators are resolved into factors.

Example V. When several fractions are to be added or subtracted, much labour can be saved by a little thought as to the arrangement of the fractions. Side multiplications can often be avoided by taking only two or three fractions at a time, so that there is no necessity to multiply together more than two binomial expressions in any one step.

Simplify $\dfrac{1}{6x-18} - \dfrac{1}{6x+18} - \dfrac{1}{x^2+9} + \dfrac{18}{x^4+81}$.

$$\text{Expression} = \frac{1}{6(x-3)} - \frac{1}{6(x+3)} - \frac{1}{x^2+9} + \frac{18}{x^4+81}$$

$$= \frac{x+3-x+3}{6(x^2-9)} - \frac{1}{x^2+9} + \frac{18}{x^4+81}$$

$$= \frac{x^2+9-x^2+9}{x^4-81} + \frac{18}{x^4+81}$$

$$= 18\left\{\frac{1}{x^4-81} + \frac{1}{x^4+81}\right\}$$

$$= \frac{36x^4}{x^8-6561}.$$

139. Equations containing Fractions.

The following examples show how to deal with equations containing algebraical fractions:

Example I.

Solve $\dfrac{3}{x-2}+\dfrac{4}{x-3}=\dfrac{7}{x-4}$.

Multiply by L. C. M. of denominators. [§ 137 (iii).]

$$3(x-3)(x-4)+4(x-2)(x-4)=7(x-2)(x-3).$$

Remove brackets

$$3x^2-21x+36+4x^2-24x+32=7x^2-35x+42.$$

Transpose and collect $-10x=-26$

$$x=2\tfrac{3}{5}.$$

Example II.

Solve $\dfrac{2x-2}{x-2}-\dfrac{x+4}{x+3}=\dfrac{3x-8}{3x-11}$.

If we were to multiply by the L. C. M. of the denominators we should have to multiply out $(2x-2)(x+3)(3x-11)$ and two similar expressions. This would involve three sums at the side, as these multiplications are too long for mental work. It is always well to avoid side work, so the following method may be adopted.

Bring each side to its own L. C. D.

$$\frac{2x^2+4x-6-x^2-2x+8}{(x-2)(x+3)}=\frac{3x-8}{3x-11}.$$

Collect the numerators

$$\frac{x^2+2x+2}{x^2+x-6}=\frac{3x-8}{3x-11}.$$

Mentally resolve into factors and see if any factor divides out. Here there is no common factor.

Multiply now by the L. C. M. of the denominators

$$(x^2+2x+2)(3x-11)=(x^2+x-6)(3x-8).$$

Remove brackets

$$\begin{aligned} 3x^3+6x^2+6x &= 3x^3+3x^2-18x \\ -11x^2-22x-22 &\quad -8x^2-8x+48. \end{aligned}$$

Collect $-5x^2-16x-22 = -5x^2-26x+48$.

Transpose and collect $10x=70$.

Divide by 10 $x=7$.

Verify L. H. S. $=\frac{12}{5}-\frac{11}{10}=\frac{13}{10}$.

R. H. S. $=1\frac{3}{10}$.

Example III.

Solve $\dfrac{2x-2}{x-2}=\dfrac{x+4}{x+3}=\dfrac{3x-8}{3x-11}$.

[When the numerators are not of lower dimensions than the denominators, equations are often simplified by dividing each numerator by its denominator until the remainder is of lower dimensions than the denominator. This division can usually be worked mentally just as $\frac{37}{8}=4\frac{5}{8}$ is done mentally, e.g. $\dfrac{4x-16}{x-2}=4-\dfrac{8}{x-2}$.]

$$\frac{2x-2}{x-2}-\frac{x+4}{x+3}=\frac{3x-8}{3x-11}.$$

Divide numerators by denominators

$$2+\frac{2}{x-2}-\left(1+\frac{1}{x+3}\right)=1+\frac{3}{3x-11}.$$

Simplify: $$\frac{2}{x-2}-\frac{1}{x+3}=\frac{3}{3x-11}.$$

Multiply by L.C.D.:

$$2(3x^2-2x-33)-(3x^2-17x+22)=3(x^2+x-6).$$

Remove brackets:

$$6x^2-4x-66-3x^2+17x-22=3x^2+3x-18.$$

Transpose and collect:

$$10x=70$$
$$x=7.$$

This agrees with Example II, but the work has been much less. This method is always shorter when the quotients obtained by division cancel.

Example IV.

Solve $\dfrac{10}{x-4}-\dfrac{5}{x-3}=\dfrac{4}{x-6}+\dfrac{1}{x-1}$.

To avoid side multiplication reduce each side to its L. C. D.

$$\frac{10x-30-5x+20}{(x-4)(x-3)}=\frac{4x-4+x-6}{(x-6)(x-1)}.$$

Collecting the numerators and multiplying out the denominators

$$\frac{5x-10}{x^2-7x+12}=\frac{5x-10}{x^2-7x+6}.$$

Dividing by $5x-10$ and multiplying across

$$x^2-7x+6=x^2-7x+12.$$

Transpose and collect: $0=12$,

which is impossible, and apparently the equation has no solution.

There is, however, an error in the working, which has already been discussed in Chap. XIII, § 108.

Examples XVI d.

Simplify:

1. $\frac{x}{y}\left(\frac{1}{x-y}+\frac{2}{x+y}-\frac{3}{x}\right)$.
2. $\frac{ab-b^2}{ab+a^2}-\frac{ab-a^2}{ab+b^2}$.
3. $\frac{1}{2-x}+\frac{1}{3-x}+\frac{2}{x-6}$.
4. $\frac{a^2-b^2}{a^2+b^2}-\frac{a^2+b^2}{a-b}$.
5. $\frac{1}{6(1+x)}+\frac{1}{6(1-x)}+\frac{1}{3(1+x^2)}$.
6. $\frac{x+y}{x^3-y^3}+\frac{x-y}{x^3+y^3}-\frac{2(x^2-y^2)}{x^4+x^2y^2+y^4}$.
7. $\frac{1}{ax-2bx}-\frac{4b}{a^2x-4b^2x}+\frac{1}{ax+2bx}$.
8. $\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$.
9. $\left(\frac{a}{a+x}+\frac{x}{a-x}\right)\div\left(\frac{a}{a-x}-\frac{x}{a+x}\right)$.
10. $\frac{p+2}{2}-\frac{p}{p+2}-\frac{p^3-2p^2}{2p^2-8}$.
11. $\left\{\frac{p}{p-1}-\frac{p}{p+1}\right\}\div\frac{1}{3}\left\{\frac{1}{p-1}-\frac{1}{p+1}+\frac{1}{1-p^2}\right\}$.
12. $\frac{x^2+6x^3-2x-5}{x^3+9x^2+19x+10}$.
13. $\frac{a-b}{a(a+b)}-\frac{a+b}{b(b-a)}-\frac{3a-b}{a^2-b^2}$.
14. $\left(\frac{x-a}{x+a}+\frac{x+a}{x-a}\right)\div\left(\frac{a-x}{a+x}-\frac{a+x}{a-x}\right)$.
15. $\frac{2x+3}{x-x^2}+\frac{5-x}{x-1}-\frac{3}{x}$.
16. $\left(x+1+\frac{8}{x-5}\right)\left(x-1-\frac{8}{x-3}\right)$.
17. $\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}-\frac{a+b}{(c-a)(c-b)}$.

18. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2-ca}{(b-c)(b-a)}+\dfrac{c^2-ab}{(c-a)(c-b)}$.

19. $\dfrac{a}{(a-x)^2}-\dfrac{3a}{2a^2-ax-x^2}+\dfrac{1}{2a+x}$.

20. $\dfrac{a-\dfrac{b^2}{a}}{\dfrac{a^3}{b^2}-\dfrac{b^2}{a}}\times\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$.

21. $\left(\dfrac{3}{2a-b}-\dfrac{2}{3a+b}\right)\left(1+\dfrac{2a}{a+b}\right)$.

22. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

23. $\dfrac{x-1}{x-2}\left\{1-\dfrac{1}{x+1}-\dfrac{2}{x^2-1}\right\}$.

24. $\left\{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right\}\div\left\{\dfrac{a^2+ax}{ax-a^2}-\dfrac{a^2-ax}{ax+x^2}\right\}$.

25. $\left\{\dfrac{x+2a}{a-2x}-\dfrac{a+2x}{x-2a}\right\}\times\left\{\dfrac{3}{2a-x}-\dfrac{1}{a-x}\right\}$.

Solve the following equations:

26. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{2}{x}$.

27. $\dfrac{1}{x}+\dfrac{3}{x-1}=\dfrac{4}{x-2}$.

28. $\dfrac{x-a}{1+b}-\dfrac{x-b}{1+a}=b-a$.

29. $\dfrac{ax}{ax-1}+\dfrac{bx}{bx-1}=2$.

30. $\dfrac{x+a+c}{x+b+c}=\dfrac{b}{a}$.

31. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}$.

32. $\dfrac{2(3-4x)}{3-x}+\dfrac{3}{1-x}=8$.

33. $\dfrac{3+2x}{1+2x}-\dfrac{5+2x}{7+2x}=1-\dfrac{4x^2-2}{7+16x+4x^2}$.

34. $\dfrac{x}{2x-1}-\dfrac{\frac{1}{2}x-\frac{1}{4}}{4x^2-1}+\dfrac{1}{2x+1}=\frac{1}{2}$.

Simplify:

35. $\dfrac{\dfrac{a+2b}{a+b}+\dfrac{a}{b}}{\dfrac{a+2b}{b}-\dfrac{a}{a+b}}\div\dfrac{\dfrac{a^2}{b^2}+\dfrac{1}{a}}{\dfrac{a}{b^2}-\dfrac{1}{b}+\dfrac{1}{a}}$.

36. $\dfrac{(a-b)^2-c^2+(a+c)^2-b^2+(c-b)^2-a^2}{(b+c)^2-a^2+(c+a)^2-b^2+(a+b)^2-c^2} \times \dfrac{a+b+c}{a-b+c}$.

37. $\dfrac{(1-x^2)(1-x^3)}{x(1+x)(1-x)^2} - \dfrac{x^3-\dfrac{1}{x^3}}{x^2+\dfrac{1}{x^2}+1}$.

38. $\dfrac{(1-x^2)(1-y^2)-4xy}{(1-x)(1-y)-2xy}$.

39. $\dfrac{x^2+xy+y^2-a(x+y)+a^2}{x+y+a} + \dfrac{x^2-xy+y^2+a(x-y)-a^2}{x-y+a}$.

40. $\dfrac{\dfrac{x^2}{x^2+y^2}}{\dfrac{x+y}{}-y} + \dfrac{\dfrac{y^2}{x^2+y^2}}{\dfrac{x-y}{}-x}$.

41. $\dfrac{(x+a)^3+(x+b)^3-(2x+a+b)^3}{(x+a)(x+b)(2x+a+b)}$.

42. $\dfrac{2y}{x+y} + \dfrac{1+\dfrac{x}{y}+\dfrac{x^2}{y^2}}{1-\dfrac{x}{y}+\dfrac{x^2}{y^2}} \times \dfrac{y^3+x^3}{y^3-x^3}$.

43. $\dfrac{1}{x-1} \div \left(\dfrac{x+\frac{1}{2}}{x-1} - \dfrac{x+1}{x-\frac{1}{2}}\right)$.

44. $\dfrac{x+y}{x+2y+\dfrac{3y^2}{x-2y}} \times \dfrac{x-y}{x^2-2xy-2y^2}$.

45. $\dfrac{1}{(1-a)(1-b)} - \dfrac{a^2}{(1-a)(a-b)} + \dfrac{b^2}{(1-b)(a-b)}$.

Solve the following equations:

46. $\dfrac{3}{x-3} + \dfrac{4}{x-4} = \dfrac{25}{x(x-7)+12}$.

47. $\dfrac{1}{6x+9} + \dfrac{2}{10x+1} = \dfrac{2}{2x+3}$.

48. $\dfrac{x-2a}{b+c-a} + \dfrac{x-2b}{c+a-b} + \dfrac{x}{a+b+c} = 0$.

49. $\dfrac{1}{(x+a)} - \dfrac{1}{x+b} - \dfrac{1}{x+a+b} + \dfrac{1}{x+2b} = 0$.

50. $\dfrac{7}{x+2} - \dfrac{10}{x-5} + \dfrac{3}{x+4} = 0$.

Examples XVI.

1. Simplify $\dfrac{a^2-2ab+b^2}{a^2+3ab-4b^2}$.

2. Simplify $\dfrac{7x+6}{2(x+2)}+\dfrac{4}{x+2}-\dfrac{7}{2}$.

3. Simplify $\dfrac{x^2-4}{x^2+3x+2}\times\dfrac{(x+1)^2}{x^2-x-12}\div\dfrac{x^2-x-2}{x^2-4x}$.

4. Solve $\dfrac{5x-(x-3)}{3x+(x-2)}=\dfrac{9}{4}$.

5. *A* had £50 more than *B*; *A* paid a third part of his money to *B*, and *B* paid back a fifth part of what he then had: the result was that *B* had £30 more than *A*. How much had each at first?

---

6. Simplify $\dfrac{a-b}{b}+\dfrac{2a}{a-b}-\dfrac{a^2}{b(a-b)}$.

7. Reduce $\dfrac{2x^2-x-1}{3x^2-x-2}$ to its lowest terms.

8. Simplify $\dfrac{2}{a+3b}-\dfrac{1}{a+2b}$, and divide the result by $a+b$.

9. Solve $\dfrac{1}{x}+\dfrac{2}{x+1}=\dfrac{3}{x+2}$.

10. A man buys oranges at the rate of p a shilling and by selling them at q pence a dozen makes a profit of r per cent. Show that $25pq-36r=3600$.

---

11. Simplify $\dfrac{2}{a-2b}+\dfrac{1}{a+2b}-\dfrac{4a}{a^2-4b^2}$.

12. Simplify $\dfrac{a^2-ab}{2bc-ab}\times\dfrac{b^2+bc}{ac-bc}\times\dfrac{2ac-bc}{ab+ac}$.

13. Reduce $\dfrac{(x-y)^2}{(x^2-y^2)(x^3-y^3)}$ to its lowest terms.

14. Solve $\dfrac{x-5a}{a+b}+\dfrac{x+b}{a-b}+1=0$.

15. If p men do the same amount of work as q boys, and a men earn £10 in 6 days, how much will b boys earn in 10 days?

16. Simplify $\frac{1}{2a+b}+\frac{1}{2a-b}=\frac{3a}{4a^2-b^2}$.

17. Solve $\frac{2x-15}{2x-6}-\frac{1}{2(x+7)}=\frac{x-7}{x+7}$, and verify the answer.

18. Simplify $\frac{3abc-3b^2c}{a-bc-b(1-c)}$.

19. Show that $\frac{1}{x-a}+\frac{1}{x-b}$ has the same value when

$$x=a+b, \text{ as when } x=\frac{2ab}{a+b}.$$

20. A can do a piece of work in x hours, B can do it in y hours. How long will A and B take to do it working together? After they have worked together for z hours, what fractional part remains to be done?

---

21. Simplify $\left(\frac{1}{x+3}-\frac{1}{2x+3}\right)\left(2+\frac{3}{x}\right)$.

22. Simplify $\dfrac{\frac{1}{a}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{b}\left(\frac{-1}{a}+\frac{1}{b}\right)}{\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{a}+\frac{1}{b}\right)^2}$.

23. Solve $\frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{x+3}-\frac{1}{x+4}$.

24. Simplify $\frac{x^3-y^3}{(x+y)^2}\times\frac{x^2+y^2}{x^2+xy-2y^2}\div\frac{x^4+x^2y^2+y^4}{x^2+3xy+2y^2}$.

25. A man bought a certain number of sheep for forty guineas. He lost one-tenth of them, and sold the rest for half a guinea a head more than he had given for them and thus gained five guineas. How many sheep did he buy?

---

26. Simplify $-7ac-\{2c(a-3b)-3a(5c-2b)\}$; and find its value when $c=\frac{ab}{a+b}$.

27. Simplify $\frac{a^2-b^2}{a^2b^2}+\frac{b^2-c^2}{b^2c^2}+\frac{c^2-a^2}{c^2a^2}$.

28. Simplify $\frac{x^2-x-6}{x^2-3x+2}\times\frac{x^2+x-6}{x^2+x-2}\times\frac{x^2-1}{x^2+6x+9}$.

29. Solve $\frac{1}{x-2} - \frac{1}{3-x} - 1\frac{1}{2} = 0$.

30. A party consists of men, women, and boys; the men are one more than one-sixth of the whole, the women two more than one-quarter of the whole, and the boys five more than one-half of the whole. How many are there altogether?

---

31. Reduce to its lowest terms $\frac{x^5+3x^4y+x^3y^2+3y^5}{x^5+3x^4y+2x^3y^2+6y^5}$.

32. Find the value of $\frac{a-p}{b-p}$ when $p = \frac{ab}{a+b}$.

33. Simplify $\frac{p}{p^2+5p+6} + \frac{15}{p^2+9p+14} - \frac{12}{p^2+10p+21}$.

34. Solve $\frac{6x+8}{2x+1} - \frac{2x+38}{x+12} = 1$.

35. Some smugglers discovered a cave which would exactly hold the cargo of their boat, viz. 13 bales of cotton and 33 casks of rum. Whilst they were unloading, a custom house cutter came in sight, so they sailed away with 9 casks and 5 bales, having filled only $\frac{3}{5}$ of the cave. How many bales would the cave hold?

---

36. If $P = q-r$, $Q = r-p$, $R = p-q$, find the value of

$$\frac{1}{QR} + \frac{1}{RP} + \frac{2}{PQ}.$$

37. Simplify $\frac{x^6+8}{x^6-8} \div \frac{x^2+2}{x^2-2}$.

38. Simplify $\frac{p}{p-q} + \frac{q}{p+q} + \frac{pq}{p^2-q^2} - \frac{q^2}{p^2+q^2}$.

39. Solve $\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}$.

40. A shepherd in time of war was robbed by a party of soldiers, who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep; another party took $\frac{1}{3}$ of what he had left and $\frac{1}{3}$ of a sheep. A third party took $\frac{1}{2}$ of what now remained and $\frac{1}{2}$ of a sheep. He had 25 sheep left; how many had he at first? Verify.

---

41. Simplify $\frac{p}{q-r} + \frac{q}{r-p} + \frac{pq-q^2}{(q-r)(r-p)}$.

42. Simplify $\dfrac{\dfrac{a}{1+a}+\dfrac{a}{1-a}}{\dfrac{1}{1-a}-\dfrac{1}{1+a}}$.

43. Find the value of $6\sqrt{ax^3}-\dfrac{2ax^3}{(a-x)^2}-a^2$ when $a=4$, $x=2$.

44. Solve $\dfrac{5x+6}{2x-4}=\dfrac{10x-7}{4x-2}$.

45. Find a fraction, such that if 4 be added to its numerator its value is $\frac{1}{2}$ and if 7 be added to the denominator its value is $\frac{1}{3}$.

---

46. Simplify $\dfrac{-2x+5}{4x^2+4x-3}-\dfrac{2x+7}{4x^2+8x+3}+\dfrac{2x-5}{4x^2-1}$.

47. Multiply $\dfrac{x}{y}-1+\dfrac{y}{x}$ by $\dfrac{x}{y}+1-\dfrac{y}{x}$.

48. Simplify $\dfrac{(x^2y^2)^{\frac{1}{2}}\left(1+\dfrac{x}{y}\right)^2}{\left(1+\dfrac{y}{x}\right)}\times\left(\dfrac{1}{x}+\dfrac{1}{y}\right)^2$.

49. Solve $\dfrac{9}{x+6}+\dfrac{15}{2x+6}=\dfrac{9}{2x+12}+\dfrac{90}{8x+24}$.

50. A does $\frac{3}{5}$ of a piece of work by himself in 30 days, and with the help of B finishes it in 10 days more. How long would the work have taken if they had worked together all the time?

---

51. What must be added to $\dfrac{4x}{(x-1)^2(x+1)}-\dfrac{x+1}{(x-1)^2}+1$ to make 2?

52. Simplify $\dfrac{1-\dfrac{2xy}{x^2+y^2}}{\dfrac{x^3-y^3}{x-y}-3xy}$.

53. When $a=1$, $b=-2$, $c=4$, $d=-3$, find the value of

$$\frac{abcd}{b^2c^2}\times\sqrt{d^2+(c-b)^2}+\frac{2cd}{3ab^2}.$$

54. Find the value of $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}+\dfrac{1}{ax-ac}$ when

$$x=\frac{b}{a}(a-b+c).$$

55. A train starts on a journey of 250 miles at the rate of x miles an hour; after going 100 miles the rate is increased by 5 miles an hour; 2 hours afterwards the rate is again increased by 5 miles an hour. Find an expression for the whole time taken.

---

56. Write two common multiples of the 4th degree in x and y of the following: $2x^3-xy-y^3$, $2x^3-3xy+y^3$. Find the L. C. M.

57. Prove that $\dfrac{y-z}{1+yz}+\dfrac{z-x}{1+zx}+\dfrac{x-y}{1+xy}=\dfrac{(y-z)(z-x)(x-y)}{(1+yz)(1+zx)(1+xy)}$.

58. A man has x sovereigns and y pence. He dies and leaves to each of his m sons one-tenth, and to each of his n daughters one-twelfth of his possessions, while his wife gets the remainder. How much does she get?

59. Simplify $\dfrac{x^2-4xy-21y^2}{x^2-9xy+20y^2}\div\dfrac{x^2-13xy+42y^2}{x^2+3xy-40y^2}$.

60. Solve (i) $\dfrac{x^3-27}{x-3}-\dfrac{x^3+27}{x+3}=6$;

(ii) $\dfrac{1}{x+2+\dfrac{1}{x+3}}=\dfrac{1}{x-2+\dfrac{1}{x-3}}$.

CHAPTER XVII

QUADRATIC EQUATIONS. GRAPHICAL SOLUTIONS

ELEMENTARY THEORY OF QUADRATICS

140. Definition. **A quadratic equation is one in which the term of highest dimensions is of the second degree.**

In dealing with quadratic equations, fractions, brackets, &c., should be removed in the ordinary way, and all the terms transposed to the left-hand side and collected. All quadratic equations are thus brought to the **standard form**, viz.

$$ax^2+bx+c=0$$

where a, b, c may be any quantities whatever not containing x.

There are several methods of solution, and it is only by continual practice that the student can tell the best method to adopt in particular cases.

141. Method I. When there are no terms of the first degree in the equation.

Example I. *Solve* $(x+2)^2 = 4x+5$.

$$(x+2)^2 = 4x+5.$$

Remove brackets $x^2+4x+4 = 4x+5.$

Transpose $x^2 = 1.$

Take square root $x = \pm 1.$

It is important to notice that whenever a square root is taken the result may be either positive or negative.

[Verify. If $x = +1$, $(x+2)^2 = 9$,
$4x+5 = 9.$
If $x = -1$, $(x+2)^2 = 1.$
$4x+5 = 1.$]

Example II. *Solve* $(x+2)^2 = 4x+6$.

Just as in Example I we get $x^2 = 2.$

Take square root $x = \pm \sqrt{2}.$

By Arithmetic the square root of 2 may be found to any required number of places, and hence the value of x may be found to any required degree of accuracy.

Example III. *Solve* $(x+a)^2 = 4x+2$.

As in Example I we get $x^2 = -2.$

Take square root $x = \pm \sqrt{-2}.$

But no real number when multiplied by itself is negative, therefore $\sqrt{-2}$ does not exist, and is said to be imaginary.

142. Method II. By factors. This is the easiest way of solving quadratics provided the factors are rational (see § 129). By using the method of § 122 quadratics can always be solved by factors, even though the roots are not rational or not real.

Example I. *Solve* $x^2-17x+60 = 0$.

$$x^2-17x+60 = 0.$$

Factorize $(x-5)(x-12) = 0.$

But if the product of two quantities is nothing, one of the quantities must be nothing. Hence

either $x-5 = 0$ or $x-12 = 0.$

$\therefore\ x = 5$ or $12.$

Example II. *Solve* $x^2-8x-13=0$.

$$x^2-8x-13=0.$$

Factorize $\quad x^2-8x+16-29=0,$

$$(x-4)^2-29=0.$$

$$(x-4-\sqrt{29})(x-4+\sqrt{29})=0.$$

$\therefore$ either $x-4-\sqrt{29}=0$ or $x-4+\sqrt{29}=0$;

i.e. $x=4+\sqrt{29}$ or $4-\sqrt{29}$.

143. Method III. By completing the square. We know that $x^2+2ax+a^2=(x+a)^2$, therefore by adding a^2 to x^2+2ax we make an exact square. This process is called **completing the square**; it has already been used in § 122 and § 142 and is of frequent occurrence.

Example I. *Solve* $x^2-8x-13=0$.

$$x^2-8x-13=0.$$

Transpose the constant to the right-hand side,

$$x^2-8x=13.$$

Add the square of half the coefficient of x to each side,

$$x^2-8x+(\tfrac{8}{2})^2=13+16.$$

Take the square root $\quad x-4=\pm\sqrt{29}.$

Transpose $\quad x=4\pm\sqrt{29}.$

If a numerical answer is required the square root may be extracted to any required number of places.

144. Method IV. By formula.

Apply Method III to the standard form,

$$ax^2+bx+c=0.$$

Divide by a and transpose the constant,

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Complete the square,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a};$$

$$\text{i.e.}\quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Take square root $\quad x+\dfrac{b}{2a}=\pm\dfrac{\sqrt{b^2-4ac}}{2a}.$

Transpose $$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

By giving the correct values to a, b, c, this formula gives the solution of any quadratic.

Example I. *Solve* $3x^2-5x-7=0$.

Here $a=3$, $b=-5$, $c=-7$;

$$\therefore\ x = \frac{5 \pm \sqrt{25+84}}{6} = \frac{5 \pm \sqrt{109}}{6}.$$

Examples XVII a.

Solve, by as many methods as possible, each of the following equations, and in each case find the sum and product of the roots. When the roots are irrational, they should be worked correct to two decimal places.

1. $(x-3)^2 = 5(5-x)-x$.
2. $(x-3)^2 = (x-2)^2$.
3. $(2x-3)^2 = (x-2)^2$.
4. $x^2-15x+56=0$.
5. $x^2+49x-660=0$.
6. $3x^2-6x+3=0$.
7. $4x^2-12x+9=0$.
8. $3x^2+7x-6=0$.
9. $x(x-2)-3(x-2)=0$.
10. $x^2(x-2)-3(x-2)=0$.
11. $x^2+x-380=0$.
12. $12x^2-7x-10=0$.
13. $3x^2+4x-5=0$.
14. $3(x^2-2) = 4(x-1\frac{1}{2})$.
15. $(x-5)(x-7) = 3$.
16. $(x-5)(x-7) = 3(x-5)$.
17. $3x^2-11x+8=0$.
18. $y^2-8y-105=0$.
19. $x^2-2ax+a^2=b^2$.
20. $x^2-7ax+12a^2=0$.
21. $3x^2+4x+5=0$.
22. $x^2-3x+1=0$.
23. $x^2-x-600=0$.
24. $2x^2+7x+6=0$.
25. $4x^2+9x+3=0$.
26. $x^2+6x-7=0$.
27. $5x^2+x-8=0$.
28. $x^2+4x-21=0$.
29. $7y^2-8y+3=0$.
30. $x^2-1{\cdot}2x+{\cdot}11=0$.

Without actually solving, write down the sum and product of the roots of the following equations:

31. $x^2-5x+3=0$.
32. $x^2+17x-301=0$.
33. $x^2+91x-1002=0$.
34. $x^2+4bx-18c=0$.
35. $3x^2-44x+77=0$.
36. $3x-4x^2=19$.

Find the simplest equations which have the following roots:

37. $x=5$ or 6.
38. $x=3$ or -2.
39. $x=-1$, or 5, or 6.
40. $x=a+b$ or $a-b$.
41. $x=17$ or $-\frac{3}{5}$.
42. $x=3\frac{1}{2}$ or $-\frac{1}{2}$.

43. Find an expression which vanishes when $x = 3$ or -4.

44. Find an expression which is equal to zero when $y = 7$ or -2.

45. Find a rational expression which equals nothing when $x = \sqrt{2}$. For what other value is this expression equal to nothing?

46. By using the Factor Theorem, show that a quadratic equation cannot have more than two roots.

47. Between what values must x lie in order that $(x-3)(x-4)$ may be negative?

48. Show that if x is a real quantity x^2-4x+6 cannot be negative.

49. Between what values must k lie in order that k^2+k-2 may be positive?

50. Find the values of α and β if $\alpha+\beta = 9$, $\alpha\beta = 18$.

Graphical Methods of Solution.

145. *Solve* $x^2-x-6 = 0$.

There are three different ways of stating the question we are trying to answer, viz.

What value of x will make $x^2-x-6 = 0$?

What value of x will make $x^2-x = 6$?

What value of x will make $x^2 = x+6$?

There are three distinct graphical methods of solving the equation corresponding to these three questions.

Graphical Method I.

Solve $x^2-x-6 = 0$.

What value of x makes $x^2-x-6 = 0$?

Draw the graph of $y = x^2-x-6$ (§ 47), and find the abscissae of the points where it cuts the axis of x.

x	-3	-2	-1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
x^2-x	12	6	2	0	0	2	6	12
y	6	0	-4	-6	-6	-4	0	6

In Fig. 27, it is seen that the graph cuts the axis of x at the points A and B; therefore at the points A and B, $y = 0$. But for the point A, $x = -2$, and for the point B, $x = 3$. Hence, when $x = -2$ or 3, $y = 0$; but $y = x^2 - x - 6$.

$\therefore$ the required roots are $x = -2$ or 3.

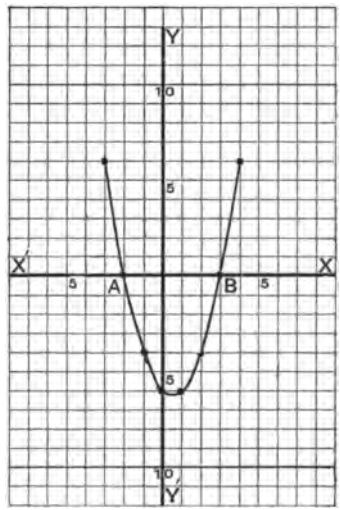

FIG. 27.

The same method can be applied in all cases whether the roots are exact integers or not. With the scale of Fig. 27 the roots can be determined to one decimal place; and by choosing a larger scale to two decimal places.

146. Graphical Method II.

Solve $x^2 - x - 6 = 0$.

What value of x makes $x^2 - x = 6$?

Draw the graph of $y = x^2 - x$ and find the abscissae of the points where it cuts the line $y = +6$.

x	-3	-2	-1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
y	12	6	2	0	0	2	6	12

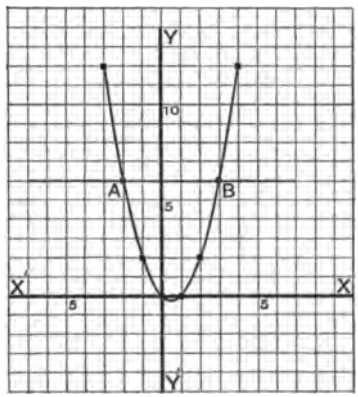

FIG. 28.

The graph, as shown in Fig. 28, cuts $y = 6$ at the points A and B whose abscissae are -2 and $+3$ respectively.

$\therefore$ required solution is $x = -2$ or $+3$.

Note i. When the graph $y = x^2 - x$ is drawn we can at once solve any equation of the form $x^2 - x - c = 0$ where c is a number, e. g.

The roots of $x^2 - x - 4 = 0$ are $x = 2{\cdot}6$ or $-1{\cdot}6$ since the graph cuts $y = 4$ at points where $x = 2{\cdot}6$ or $-1{\cdot}6$.

Note ii. Since the graph does not go lower than $y = -{\cdot}2$ (approximately), it is seen that if c is less than $-{\cdot}5$, there are no real values for x. An accurately drawn graph on a larger scale would show exactly the least value of y, but with this graph we are not justified in saying more than that the least value of y is about $-{\cdot}2$ or $-{\cdot}3$.

147. Graphical Method III.

Solve $x^2-x-6=0$.

What value of x makes $x^2=x+6$?

Draw the graphs of $y=x^2$ and $y=x+6$, and find the abscissae of the points of intersection.

(i) Graph of $y=x^2$:

x	-3	-2	-1	0	1	2	3	4
y	9	4	1	0	1	4	9	16

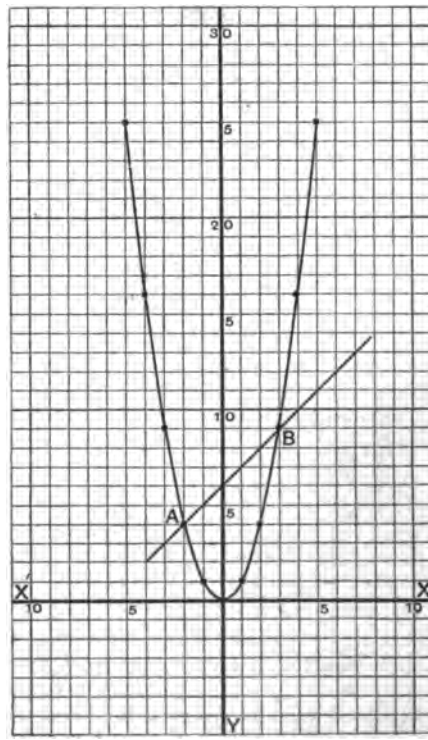

FIG. 29.

(ii) Graph of $y = x + 6$.

The equation is of first degree, $\therefore$ the graph is a straight line.

x	0	-3	$+3$
y	6	3	$+9$

Fig. 29 shows that the two graphs cut at the two points A and B, where $x = -2$, $x = +3$ respectively.

Hence required solution is $x = -2$ or $+3$.

Note. The graph of $y = x^2$ should be drawn accurately on a fairly large scale, and can then be used for solving any quadratic equation. The straight line need not be actually drawn, a piece of thread can be stretched across the graph of $y = x^2$, and the abscissae of the points of intersection can easily be read.

Or a piece of cardboard or celluloid can be cut to the shape of the graph $y = x^2$; with this the graph can be reproduced whenever required.

Examples XVII b.

Solve the following equations by each of the three graphical methods, the answers being correct to one decimal place:

1. $x^2 - 4x - 5 = 0$.
2. $x^2 - 3x - 5 = 0$.
3. $x^2 + 6x + 8 = 0$.
4. $x^2 + 9x + 3 = 0$.
5. $x^2 - 5x + 8 = 0$.
6. $x^2 - 12x = -36$.
7. $(x-3)(x+4) = 15$.
8. $4x^2 - 5x - 6 = 0$.
9. $3x^2 - 11x + 14 = 0$.
10. $7x^2 - 2x - 1 = 0$.

By drawing graphs, find approximately at least one root of each of the following equations:

11. $x^3 + 2x^2 - 3x + 4 = 0$.
12. $x^3 - 3x + 2 = 0$.
13. $x^3 - 7x^2 + 3 = 0$.

Solve by one of the graphical methods as correctly as you can:

14. $x^2 - 2{\cdot}3x + 1{\cdot}1 = 0$.
15. $x^2 + 3{\cdot}4x - 2{\cdot}3 = 0$.
16. $x^2 - 4{\cdot}1x = 6{\cdot}7$.
17. $3x^2 - 11x + 7{\cdot}2 = 0$.

ELEMENTARY THEORY OF QUADRATICS

148. Every quadratic equation has two roots and not more than two different roots.

Let $ax^2 + bx + c = 0$ be the quadratic equation. It is shown in

§ 144 that this equation has two roots. If possible, let the equation have three different roots, a, β, γ, then

$$a\alpha^2+b\alpha+c = 0, \quad \text{(i)}$$

$$a\beta^2+b\beta+c = 0, \quad \text{(ii)}$$

$$a\gamma^2+b\gamma+c = 0. \quad \text{(iii)}$$

Subtract ii from i $\quad a(\alpha^2-\beta^2)+b(\alpha-\beta) = 0.$

We may divide by $\alpha-\beta$ since α and β are supposed different;

$$\therefore \quad a(\alpha+\beta)+b = 0. \quad \text{(iv)}$$

Similarly, by subtracting iii from ii, we obtain

$$a(\beta+\gamma)+b = 0. \quad \text{(v)}$$

Subtract v from iv $\quad a(\alpha-\gamma) = 0;$

but $\alpha-\gamma$ is not zero, $\quad \therefore \quad a = 0.$

Substitute in iv $\quad b = 0.$

Substitute in i $\quad c = 0.$

Hence the equation cannot be satisfied by three different values unless all the coefficients are zero; in that case the equation is an identity.

Example. *Prove the identity*

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)} = 1.$$

The left-hand side is a quadratic expression; the equation is satisfied by $x = a$, $x = b$, $x = c$, and the equation is, therefore, an identity.

149. The sum of the roots of any quadratic equation is equal to the coefficient of x with the sign changed, divided by the coefficient of x^2; and the product of the roots is equal to the constant term divided by the coefficient of x^2.

Method I.

Consider the standard form

$$ax^2+bx+c = 0.$$

Divide by a $\quad x^2+\frac{b}{a}x+\frac{c}{a} = 0.$

To solve we try to resolve this into factors; in order to resolve into factors, we try to find two quantities whose product is $\frac{c}{a}$ and whose algebraic sum is $\frac{b}{a}$. Suppose h and k are these quantities,

then $$(x+h)(x+k) = 0;$$

$$\therefore \quad x = -h \quad \text{or} \quad -k;$$

and sum of roots $= -(h+k) = -\frac{b}{a}$,

product of roots $= hk = \frac{c}{a}$.

Method II.

By the formula of § 144 the roots of

$$ax^2+bx+c = 0$$

are given by $$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Hence, if α and β are the roots,

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a},$$

$$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a};$$

$$\therefore \quad \alpha+\beta = \frac{-2b}{2a} = -\frac{b}{a},$$

$$\alpha\beta = \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2} = \frac{c}{a}.$$

If the quadratic equation be written

$$x^2+px+q = 0,$$

we have $$\alpha+\beta = -p, \ \alpha\beta = q.$$

By means of these relations between the roots, questions can often be answered without actually solving the equation.

Example. *Find the sum of the squares of the roots of*

$$x^2-17x-19 = 0.$$

Let α and β be the roots, then

$$\alpha+\beta = 17,$$
$$\alpha\beta = -19.$$
$$\therefore\ \alpha^2+\beta^2 = (\alpha^2+2\alpha\beta+\beta^2)-2\alpha\beta$$
$$= 17^2-(-38)$$
$$= 289+38$$
$$= 327.$$

150. Nature of the roots of a quadratic equation.

By the formula the roots of $ax^2+bx+c=0$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2}.$$

(i) If $b^2-4ac > 0$, then $\sqrt{b^2-4ac}$ is a real quantity, and the roots are real and unequal.

(*a*) If b^2-4ac is an exact square, $\sqrt{b^2-4ac}$ is rational, and the roots are real, unequal, and rational.

(*b*) If b^2-4ac is not an exact square, $\sqrt{b^2-4ac}$ is irrational, and the roots are real, unequal, and irrational.

(ii) If $b^2-4ac=0$, then $\sqrt{b^2-4ac}=0$, and the roots are real and equal.

(iii) If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ is imaginary, and the roots are imaginary and unequal.

These rules are very useful, as they enable us to choose the best method of solving an equation. If, for instance, mental work shows us that b^2-4ac is negative, it is no use to try a graph method, and there is often no need to solve the equation at all; if b^2-4ac is not an exact square, it is no use to try to solve by the factor method, and the formula method or a graph method should be adopted at once.

151. Equal roots. The student may have some difficulty in understanding what is meant by equal roots. If, for instance, we have to solve $x^2-6x+9=0$, what do we gain by saying that there are two roots equal to 3 instead of that there is one root equal to 3? It is quite clear that

there are two factors $x-3$, so that we are justified in saying that there are two roots, but what is the use of this idea of two equal roots? The answer to this depends on the original question from which the equation was derived, e. g. find where the graphs represented by the two equations $x^2+y^2=18$, $x+y=6$ meet.

It has been shown in § 64 that, in order to find where two curves meet, we must treat their equations as simultaneous equations and find the resulting values of x and y.

$$x^2+y^2=18, \qquad \text{(i)}$$

$$x+y=6. \qquad \text{(ii)}$$

From ii $\quad y=6-x.$

Substitute in ii then $\quad x^2+(6-x)^2=18;$

i.e. $\quad 2x^2-12x+36=18,$

i.e. $\quad x^2-6x+9=0, \qquad \text{(iii)}$

i.e. $\quad x=3$ or $3.$

Substitute in ii $\quad y=3$ or $3.$

Hence the two curves meet twice at the point (3, 3).

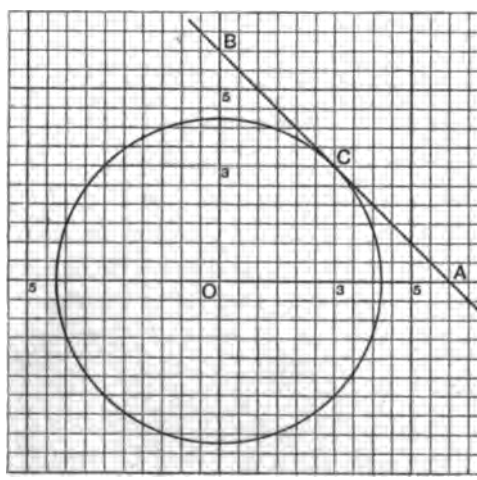

Fig. 80. Scale : 2 sides = 1 unit.

The two curves are shown in Fig. 30, where the circle is the graph of $x^2+y^2=18$ and the straight line AB is the graph of $x+y=6$. It is seen that the line AB just meets the circle at the point C where $x=3$, $y=3$, but it does not cut the circle. When two curves or lines meet but do not cut they are said to touch. This is sometimes expressed thus :

Two lines, whether straight or curved, which meet in two coincident points are said to touch one another. Hence we see that since the equation iii has two equal roots, the graphs of equations i and ii touch one another.

Examples XVII c.

If α and β are the roots of $x^2+px+q=0$, find the values of:

1. $(\alpha-\beta)(\beta-\alpha)$. 2. $\alpha^2-\beta^2$. 3. $\alpha^3-\beta^3$. 4. $\alpha^4+\beta^4$.

Without actually solving, find (i) the sum of the roots, (ii) the difference of the roots, (iii) the sum of the squares of the roots of each of the following equations:

5. $x^2-7x+8=0$. 6. $3x^2-4x+16=0$.
7. $2x^2-3x-5=0$. 8. $x^2+21x+6=0$.
9. $x^2-2ax+c=0$. 10. $x^2+2ax+a^2=b^2$.

Form the simplest equations whose roots are

11. $2+\sqrt{5}$, $2-\sqrt{5}$. 12. 3 and -8.
13. $3-\sqrt{-7}$, $3+\sqrt{-7}$. 14. $-2+\sqrt{13}$, $-2-\sqrt{13}$.

15. Find the simplest rational equation that is satisfied by $x=-4-\sqrt{5}$. What are the other roots of the equation?

16. Show that the roots of $x^2-(1+k)x+k=0$ are rational.

17. Prove that the roots of $2x^2-2(1+k)x+k=0$ are real provided k is real.

18. By solving the equation $x^2-4x+5=k$, show that if x is real x^2-4x+5 cannot be less than 1.

19. Draw the graph of $y=x^2-3x+4$. Find algebraically where it is cut by the graph of $y=mx$. For what values of m are these two points coincident? Interpret your result geometrically.

Examples XVII.

Solve the following equations:

1. $4x - \frac{14-x}{x+1} = 14.$

2. $3x - \frac{3x-10}{9-2x} = 2 + \frac{6x^2-40}{2x-1}.$

3. $\frac{x}{5+x} + \frac{7}{6-4x} = \frac{11x}{11x-8}.$

4. $x^4 - 10x^2 + 9 = 0.$

5. $x^2 - 4\sqrt{x} - 5 = 0$, and verify. $(\sqrt{x}+1)(\sqrt{x}-x+\sqrt{x}-5) = 0.$

6. $\frac{x-1}{x+1} + \frac{x+3}{x-3} = 2\left(\frac{x+2}{x-2}\right).$ (Cf. §197.)

7. $x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 6.$

8. $x^2 + \frac{1}{x^2} + 2x + \frac{2}{x} = 33.$

9. $\frac{7}{x^2+4x} + \frac{21}{3x^2-8x} = \frac{22}{x}.$

10. $(x^2-4x)^2 - 6(x^2-4x) + 5 = 0.$

11. $px^2 - qx + r = 0.$

12. $\frac{2x}{x-1} + \frac{3x-1}{x+2} - \frac{5x-11}{x-2} = 0.$

13. $x(x+b) + a(a-b) = 2ax.$

14. $(a-b)x^2 - (c-a)x + (b-c) = 0.$

15. $\frac{1+2x-3x^2}{1-2x+3x^2} = \frac{3-2x+x^2}{3+2x-x^2}.$

16. $(2x^2+3x-1)(2x^2+3x-2) = 156.$

17. $x(x+3) + a(a-3) = 2(ax-1).$

18. $\frac{5x+4}{5x-4} + \frac{5x-4}{5x+4} = 2\frac{1}{8}.$

19. $(x+1)^4 - (x-1)^4 = 14x.$

20. $156x - x^3 = 5184.$

By graphical methods obtain the roots of the following equations correct to two decimal places:

21. $4x^3 - 5x + 1 = 0.$

22. $3x^3 - 5x = 6.$

23. $3x^3 - 4x - 2 = 0.$

24. $3x^2 - 2{\cdot}34x = 3{\cdot}45$.

25. $x + \frac{1}{x} = 3$.

26. One of the roots of the equation $2x^3 - 9x^2 + 4x + 12 = 0$ is $x = 2$. Find the others.

27. Prove that, if either of the roots of a quadratic with rational coefficients is rational, they are both rational. Form the quadratic, one of whose roots is $-3 + \sqrt{5}$.

28. Prove that $6x^2 - 29x + 35$ is positive when $x = 2\frac{1}{2}$, negative when $x = 2\frac{1}{3}$, and again positive when $x = 2\frac{3}{4}$. For what values of x does it vanish? Illustrate by a graph.

29. The sum of a certain quantity and its reciprocal is 2·05. Find it.

30. If one root of a quadratic equation is known, state a simple way of finding the second root.

Solve $x(x-b) = a(a-b)$.

31. Write down the equation whose roots are 3, -4, $\frac{2}{3}$; and reduce it to its simplest form.

32. Prove that if x is real $x^2 + 4x + 11$ cannot be less than 7.

33. The difference of the roots of the equation $x^2 + ax + 6 = 0$ is 6. Find the value of a to two decimal places.

34. Find an expression which exceeds $2x^2 + 17x - \frac{1}{3}$ by the quantity by which it is exceeded by $340x^2 - 19x - 605$. For what values does the expression vanish?

35. What quantity may be substituted for a in $(a-3)(a-4)$ without altering its value?

36. By drawing graphs of x^3 and of $3x + 1$, find approximately the roots of $x^3 - 3x - 1 = 0$.

37. Without solving the equation $x^3 - 3x - 5 = 0$, find the sum of the cubes of the roots.

38. If α and β are the roots of $x^2 - 5x + 18 = 0$, find the equation whose roots are 3α and 3β.

39. By drawing a graph, or otherwise, find the least possible value for $x^2 + x - 2$, if x is a real quantity.

40. Draw the graphs of $x^2 + x - 2$ and of $3x + 6$. What equation can be solved by the use of these two graphs? What are its roots?

41. If $x^2 + 3x + 4 = y$, what value of y will give equal roots for x? Illustrate by a graph.

42. Draw the graph of $y = x^2 - 3x + 4$. What is the value of m if the graph of $y = mx$ touches the preceding graph?

43. A regiment of 1000 men is drawn up in oblong form so that the difference of the numbers in the two adjacent sides is 117. How many men were there in the long side of the oblong?

44. A farmer bought a certain number of oxen for £240, and after losing 3 sold the remainder for £8 a head more than they cost him, thus gaining £59. How many did he buy?

45. A and B distribute £60 each among a certain number of persons. A relieves 40 persons more than B, but B gives to each 5*s.* more than A. How many persons did B relieve?

46. A person bought for one crown as many pounds of sugar as were equal to half the number of crowns he spent. In selling the sugar he received for every 25 lb. of sugar as many crowns as the whole had cost him, and received on the whole 20 crowns. How many crowns did he spend, and what did he give for a pound?

47. A person bought some sheep for £72, and found that if he had bought 6 more for the same money he would have paid £1 less for each. How many did he buy?

48. A labourer dug two trenches, one of which was six yards longer than the other, for £17. 16*s.*, and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

49. A company at a restaurant had £8. 15*s.* to pay. Two left without paying their share, and each of those who were left had 10*s.* more to pay to make good the deficit. How many were there at first?

50. A regiment can be formed into a hollow square, three deep. With the addition of 25 men a solid square can be formed; the number of men in each side of the solid square is greater by 22 than the square root of the number of men in each side of the hollow square. How many men are there in the regiment?

CHAPTER XVIII

GRAPHS OF EQUATIONS OF THE SECOND AND HIGHER DEGREE. APPLICATIONS

152. The graph of any algebraic function of a single variable can be obtained by plotting a number of points as explained in § 47. In this chapter we shall give examples of the graphs that commonly occur, and shall show how the labour of plotting a large number of points may in some cases be considerably lessened.

153. It has been pointed out in § 49 that the graph of an equation of the first degree is a straight line. Care must be taken

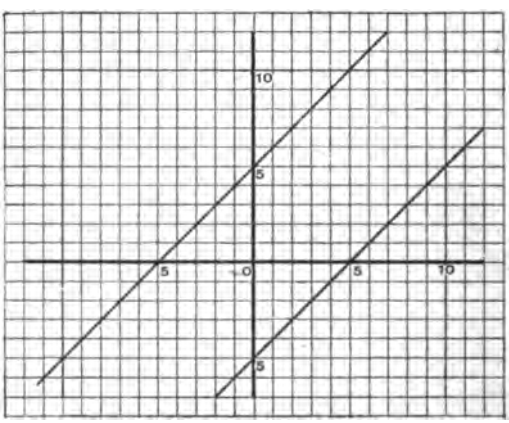

Fig. 31.

that the equation is cleared of all algebraical fractions before its degree is ascertained; for instance,

$$y = \frac{2}{x} + 3, \quad y - 3 = \frac{4}{x-5},$$

are not of the first degree and the graphs are not straight lines.

If, when all the terms of an equation are transposed to the left-hand side, the left-hand side can be resolved into a number of factors of the first degree, the graph consists of an equal number of straight lines.

Example. *Draw the graph of* $x^2-2xy+y^2 = 25$.

Equation is $(x-y)^2-5^2 = 0$;

i.e. $(x-y-5)(x-y+5) = 0$.

This equation is satisfied if either $x-y-5 = 0$, or $x-y+5 = 0$.

The graph, therefore, consists of the graphs of these two equations, that is, of the two straight lines shown in Fig. 31.

154. Equations of second degree.

Type I. $y = ax^2$.

Notice (i) That the graph passes through the origin.

(ii) That x^2 must be positive. Hence, if a is positive, y is

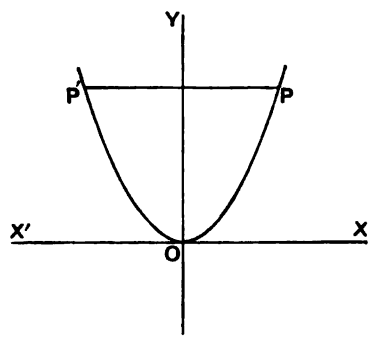

FIG. 32.

positive; if a is negative, y is negative; the curve, therefore, lies entirely above or entirely below the axis of x.

(iii) That y is unchanged, if the sign of x is changed. If the graph passes through a certain point P, it will, therefore, pass through another point P' such that PP' is parallel to the axis of x and bisected by the axis of y. We express this by saying that the graph is **symmetrical** with respect to the

axis of y. We have already had many examples of this graph which is of the form shown in Fig. 32. This curve is called a **parabola.**

Example. *Draw the graph of* $y = -3x^2$.

(i) The graph passes through the origin.

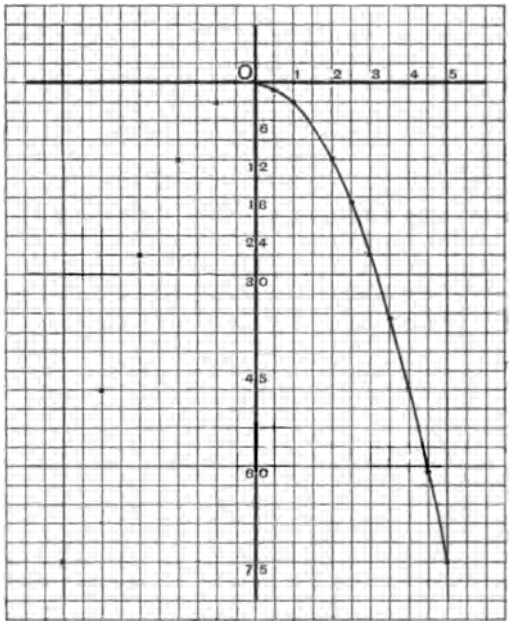

FIG. 33. Scale: 2 sides = 1 for x; 1 side = 3 for y.

(ii) Since x^2 is positive y must be negative, and the curve is entirely below the axis of x.

(iii) Since there are no odd powers of x, the graph is symmetrical with regard to the y-axis.

x	0	1	2	3	4	5	·5	2·5	3·5	4·5
x^2	0	1	4	9	16	25	·25	6·25	12·25	20·25
y	0	−3	−12	−27	−48	−75	−·75	−18·75	−36·75	−60·75

Since y soon gets large compared with x, a smaller scale must be used for y as shown in Fig. 33.

When the first six points are plotted, it is seen to be advisable to plot some intermediate points before drawing the curve. When the portion of the graph corresponding to positive values of x has been drawn, the portion corresponding to negative values can be filled in by plotting as many symmetrical points as are necessary, no further calculation is required.

The point where the parabola crosses the line of symmetry is called the *vertex*.

155. Type I (extended). $y = ax^2+bx+c$.

Several equations of this form have already occurred in connexion with the graphical solution of quadratic equations. The following way of dealing with them is instructive.

To draw the graph of $y = 3x^2-6x+5$.

Note that the graph is not symmetrical with regard to either axis and does not pass through the origin.

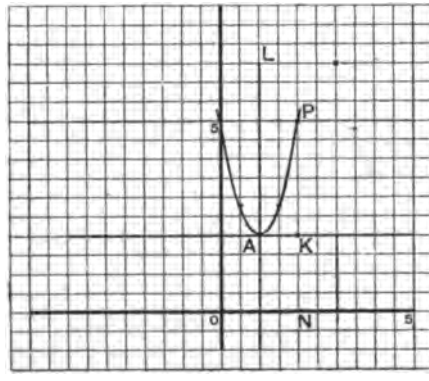

FIG. 34. Scale: 2 sides = 1 for x and y.

Rearrange the equation as follows:

$$y = 3(x^2-2x)+5 \qquad \text{(i)}$$
$$= 3(x-1)^2+2,$$
$$\text{or} \quad (y-2) = 3(x-1)^2. \qquad \text{(ii)}$$

In Fig. 34 A is the point $x = 1, y = 2$; P is any point on the

required graph so that the length of PN is y and of ON x. It follows that the length of PK is $y-2$ and of AK $x-1$. Hence equation ii asserts that $PK = 3AK^2$. That is, if A had been the origin and AK the axis of x, the equation of the curve would have been $y = 3x^2$.

But this is the form discussed in the preceding section, therefore the curve is a parabola with A as vertex and AL as line of symmetry.

In the same way it may be shown that the graph of any equation of the form $y = ax^2+bx+c$ is a parabola.

In actually plotting the graph, it is better to use the old method; the method of this section is used when one wishes to avoid actually drawing the curve.

Examples XVIII a.

Without actually drawing the graphs, state which of the following equations represent parabolas, and give all the information you can about the parabolas.

1. $y = -x^2$. 2. $y = 3x^2-5$. 3. $x = 4y^2$.
4. $x^2-y^2 = 4$. 5. $x-2y^2 = 4y$. 6. $4x-3 = 2x-3y^2$.
7. $(x-2)^2 = 3y^2-8y$. 8. $y^2 = 4x$. 9. $y^2+4x = 3$.

10–18. Draw the graphs of Nos. 1-9.

156. Type II. $xy = a$.

Take as an example $xy = 6$.

(i) The graph is not a straight line since xy is of two dimensions, nor does it pass through the origin.

x	0	±1	±2	±3	±4	±5	±6	±12
y	∞	±6	±3	±2	$\pm1{\cdot}5$	$\pm1{\cdot}2$	±1	$\pm{\cdot}5$

Since xy is positive, like signs must be taken together.

If $x = 0$, y equals the result of dividing 6 by nothing, an operation which cannot be performed. Consider what y becomes as x decreases.

x	·5	·25	·2	·1	·01	·001
y	12	24	30	60	600	6000

As x approaches the value nothing, $\frac{6}{x}$ continually increases, and can be made greater than any number that can be thought of, by taking x sufficiently small. Such a large number is called **infinity** and is denoted by the symbol ∞. The same argument can be

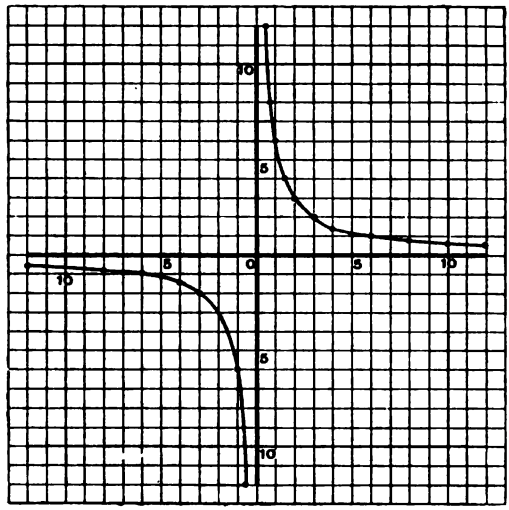

FIG. 35.

used for any number a. The result is sometimes stated concisely as follows: when $x = 0$, $\frac{a}{x} = \infty$.*

The graph of $xy = 6$ is shown in Fig. 35. This curve is known as the **hyperbola**. It consists of two distinct branches; and both branches get continually nearer and nearer to the axes but never actually meet them.

* Up to the present we have always used the symbol 0 to denote absolutely nothing, that is, the difference between two exactly equal numbers. It is here used to denote a very small number, so small that for practical purposes it may be regarded as nothing. The statement given above really means 'when x is very small, $\frac{a}{x}$ is very large.'

A straight line to which a curve gets continually nearer and nearer but which it never actually meets is called an **asymptote** of the curve.

157. Type II (extended). $xy+ax+by+c=0$.

Take as an example $xy-2x+3y-18=0$.

Factorize the variable part, adding whatever constant is necessary to complete the factorization.

$$x(y-2)+3(y-2)=12;$$

i.e. $$(x+3)(y-2)=12.$$

Then by the same kind of argument as for Type I (extended),

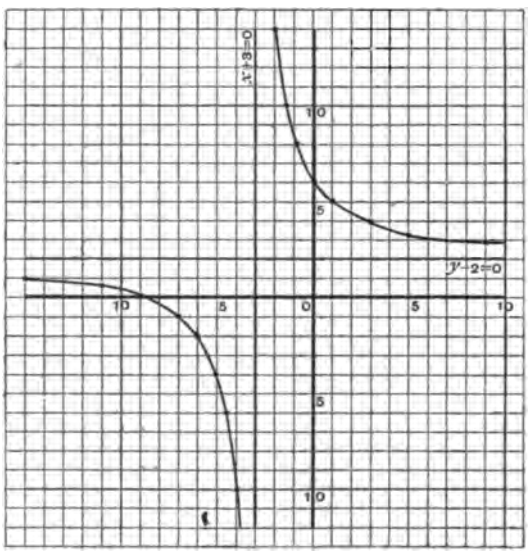

FIG. 36.

this is seen to be the equation of a hyperbola having the graphs of $(x+3)=0$ and $(y-2)=0$ for asymptotes. The graph should then be plotted as follows:

x	-3	-2	$-1{\cdot}5$	-1	0	1	3	5	9	∞
$(x+3)$	0	1	$1{\cdot}5$	2	3	4	6	8	12	∞
$(y-2)$	∞	12	8	6	4	3	2	$1{\cdot}5$	1	0
y	∞	14	10	8	6	5	4	$3{\cdot}5$	3	2

There are a similar set of negative values of $(x+3)$ and $(y+2)$ which are easily plotted by treating the asymptotes as axes and the values of $x+3$ and $y-2$ as co-ordinates.

The graph is shown in Fig. 36.

This graph can also be drawn by the following method:

$$xy-2x+3y-18=0.$$

Rearrange $$y(x+3)=2x+18$$

$$y=\frac{2(x+3)+12}{x+3}$$

$$=2+\frac{12}{x+3}.$$

Tabulate in the usual way:

x	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
$\frac{12}{x+3}$	$-2{\cdot}4$	-3	-4	-6	-12	∞	12	6	4	3	$2{\cdot}4$
y	-4	-1	-2	-4	-10	∞	14	8	6	5	$4{\cdot}4$

If $x=-3-$ a small quantity, $y=-$ a large quantity;

if $x=-3+$ a small quantity, $y=+$ a large quantity.

Examples XVIII b.

Trace the graphs of the following equations:

1. $xy=-6$.
2. $x(y-3)=9$.
3. $(x-1)(y+2)=-8$.
4. $xy-3x-4y=0$.
5. $3xy=(x-1)(y-2)$.
6. $2x+5xy=34$.
7. $(x-3)(y-2)=0$.
8. $(x-3)(y-2)=1$.
9. $(2x-3y+1)(3x-4y+2)=0$.
10. $(3x+2y-1)(3x-2y-2)=0$.
11. $2x^2+xy-y^2-3y-2=0$.
12. $6x^2-5xy-4y^2+4x-9y-2=0$.
13. $y=\frac{1}{3}\left\{\frac{30}{x-5}+23-2x\right\}$ from $x=0$ to $x=5$. Find where $y=2x$.

158. Equation of second degree.

Type III. $x^2+y^2=a^2$.

Consider $x^2+y^2=25$.

(i) Since no odd powers occur, the curve is symmetrical with regard to both axes; only positive values need be calculated, the other values being filled in by symmetry.

(ii) Since $y^2 = 25 - x^2$, and y^2 cannot be negative,

$\therefore$ x cannot be greater than 5 or less than -5.

Similarly y cannot be greater than 5 or less than -5.

x	0	1	2	3	4	5
x^2	0	1	4	9	16	25
$25 - x^2$	25	24	21	16	9	0
y	5	4·9	4·6	4	3	0

(iii) Also since x and y can be interchanged in the equation

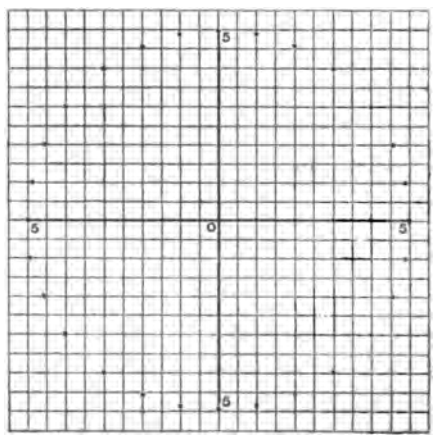

FIG. 37. Scale: side of square = ·5.

without altering the equation, we may interchange the values of x and y, thus obtaining $x = 4·9$, $y = 1$ and $x = 4·6$, $y = 2$, &c.

These values are plotted in Fig. 37 and the symmetrical points also are shown, thus giving 28 points on the graph. It is obvious that the graph is a circle with radius 5 and centre at the origin. By using compasses the graph can be drawn accurately.

159. Equations of second degree, Type III (extended).

$$x^2+y^2+ax+by+c=0.$$

By using Pythagoras' Theorem, viz. 'The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the sides containing the right angle', we can prove that the graph of an equation of the form

$$x^2+y^2+ax+by+c=0$$

is a circle and can show how the centre and radius may be determined.

circle: $a^2+b^2>4c$.
point-⊙: $=4c$.
imaj.-⊙: $<4c$.

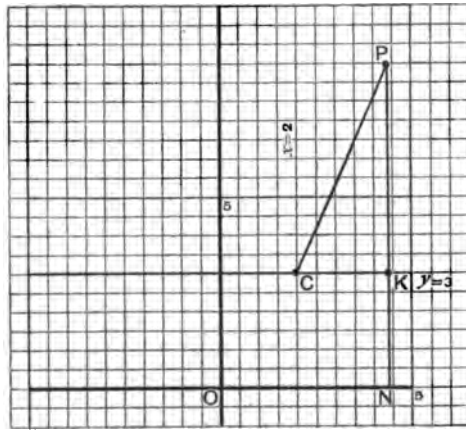

FIG. 38. Scale: side of square = .5.

Take as an example $x^2+y^2-4x-6y-23=0$. (i)

Rearrange $x^2-4x+y^2-6y=23$.

Complete the two squares by adding $4+9$ to each side,

$$(x-2)^2+(y-3)^2=36. \quad \text{(A)}$$

In Fig. 38 the graphs of $x=2$ and $y=3$ are drawn; C is the point (2, 3), and P is any point on the graph of equation (i).

Then the length of PN is y and of PK $y-3$;

also, the length of ON is x and of CK $x-2$.

Hence equation A gives the geometrical statement,

$$CK^2+PK^2=36.$$

But by Pythagoras' Theorem $CK^2 + PK^2 = CP^2$;

$$\therefore \quad CP^2 = 36, \quad \text{i.e. } CP = 6.$$

Hence the point P, i.e. any point on the graph, is at a constant distance 6 from the fixed point C.

That is, the graph of equation (i) is a circle with centre (2, 3) and radius 5.

In the same way it can always be shown that If the coefficients of x^2 and y^2 are the same both in magnitude and sign, and there is no term containing xy, then the graph of the equation is a circle.

160. Equation of second degree.

Form V $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

Example I. $\quad \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

No odd powers occur, therefore the graph is symmetrical with regard to both axes and we need calculate only positive values.

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}, \quad \text{but } \frac{y^2}{4} \text{ cannot be negative;}$$

$$\therefore \quad \frac{x^2}{9} \geqslant 1;$$

$$\therefore \quad x \geqslant 3 \text{ and } \leqslant -3.$$

Similarly $\quad y \geqslant 2$ and $\leqslant -2.$

x	0	1	2	3	2·6	2·9
$\frac{x^2}{9}$		·11	·44	1	·75	·94
$1-\frac{x^2}{9}$	1	·89	·55	0	·25	·06
y^2	4	3·55	2·22	0	1	·25
y	2	1·9	1·5	0	1	·5

Plotting these points and the symmetrical points, the graph enclosed in the rectangle *PQRS* (Fig. 39) is obtained.

This curve is called an **ellipse.**

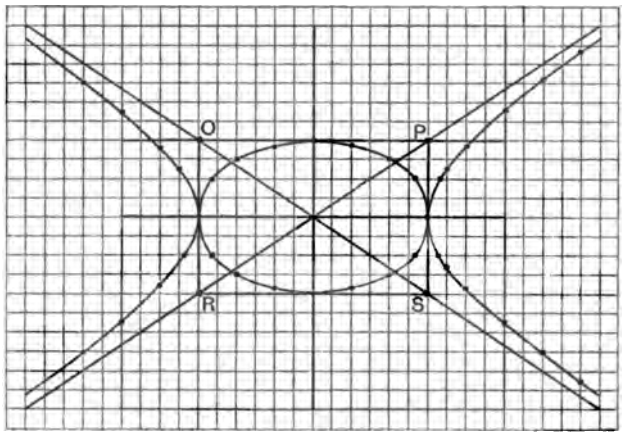

FIG. 39. Scale: side of square = ·5.

Example II. $\frac{x^2}{9} - \frac{y^2}{4} = 1.$

No odd powers occur, therefore the graph is symmetrical with respect to both axes.

$$\frac{x^2}{9} = 1 + \frac{y^2}{4} \quad (y \text{ may have any value});$$

$$\frac{y^2}{4} = \frac{x^2}{9} - 1 \quad (x^2 \text{ cannot be less than } 9).$$

x	3	4	5	6	7	8	9	10	3·5	3·3	3·1
$\frac{x^2}{9} - 1$	0	·77	1·77	3	4·44	6·11	8	10·11	·36	·25	·06
y^2	0	3·11	7·11	12	17·77	24·44	32	40·44	1·44	1	·25
y	0	1·8	2·7	3·5	4·2	4·9	5·7	6·4	1·2	1	·5

These points and the symmetrical points are also plotted in Fig. 39. The graph is seen to be a hyperbola having the diagonals of the rectangle $PQRS$ as asymptotes.

Examples XVIII c.

Draw the graphs of the following equations:

1. $x^2 = y^2 - 2(y^2 - 72)$.
2. $(x-3)^2 + (y+4)^2 = 49$.
3. $x^2 - 2x + y^2 + 6y = 26$.
4. $(3-x)^2 + (y-3)^2 = 9$.
5. $x^2 + y^2 - 4x + 4y + 8 = 0$.
6. $x^2 - y^2 = 16$.
7. $9x^2 + 16y^2 = 144$.
8. $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{16} = 1$.
9. $\frac{(x+2)^2}{16} - \frac{(y-2)^2}{9} = 1$.
10. $x^2 - y^2 + x + 3y - 2 = 0$.

Solve graphically:

11. $x^2 + y^2 = 25$, $x + y = 1$.
12. $x^2 + y^2 = 10$, $x + 4y = 7$.
13. $x^2 + y^2 - 6x + 8y = 11$, $x^2 + y^2 = 9$.
14. $x^2 + y^2 - 4x + 6y = 12$, $4x + 3y + 26 = 0$.
15. $\frac{x^2}{4} + \frac{y^2}{25} = 1$, $y - 2x = 1$.

161. Graph of equation of third degree.

The graphs of equations of degrees higher than the second must be obtained by ordinary plotting. The graph of $y = x^3$ should be drawn carefully; it is useful and is a good example of one type of cubic curve (i. e. graph of equation of third degree). We give here as an example the graph of a more complicated equation.

Example. *Draw the graph of* $y = x^3 - 2x^2 - x + 2$.

x	-3	-2	-1	0	1	2	3	-1.5	$-.5$	$.5$	1.5	2.5
x^3	-27	-8	-1	0	1	8	27	-3.37	$-.13$	$.13$	3.37	15.6
$-2x^2$	-18	-8	-2	0	-2	-8	-18	-4.5	$-.5$	$-.5$	-4.5	-12.5
$-x$	$+3$	$+2$	$+1$	0	-1	-2	-3	$+1.5$	$+.5$	$-.5$	-1.5	-2.5
y	-40	-12	0	2	0	0	8	-4.37	1.87	1.13	$-.63$	2.6

This graph is shown in Fig. 40. Notice that it has two turning points A and B.

Applications of Graphs.

162. Examples have already occurred showing how graphs may be used to solve simple simultaneous equations

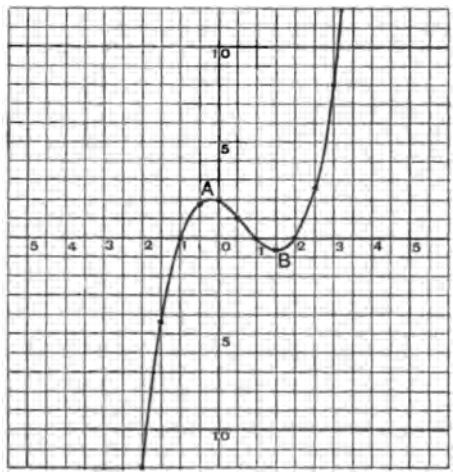

FIG. 40. Scale: 2 sides = 1 for x; 1 side = 1 for y.

and quadratic equations. Similar methods may be employed to solve equations of higher degree.

Example I. *Solve* $x^3-2x^2-x-2=0$.

This may be solved by finding where the graph of

$$y=x^3-2x^2-x-2$$

cuts the axis of x. Fig. 40 shows that the required roots are

$$x=-1,\ 1,\ \text{or}\ 2.$$

Example II. *Solve* $x^3-3x^2+2x+4=0$.

The method of Example I might be employed, or the following method.

Since the equation is equivalent to

$$x^3 = 3x^2 - 2x - 4,$$

we can solve by finding where the graphs of

$$y = x^3 \text{ and } y = 3x^2 - 2x + 4$$

intersect.

These graphs are shown in Fig. 41, where it is seen that the only real solution is $x = -.8$.

Example III. *Solve* $x^2 + y^2 - 6x + 4y = 16,$ (i)

$$xy - 3x + 2y = 6.$$ (ii)

Draw the graphs of these two equations and find the co-ordinates of their points of intersection.

Equation i is $(x-3)^2 + (y+2)^2 = 29.$

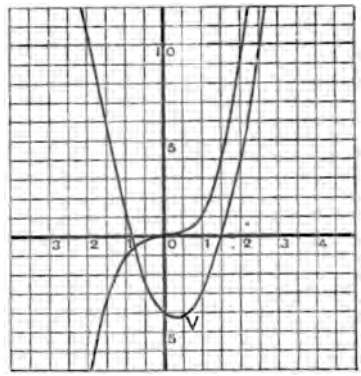

Fig. 41.

The graph is a circle whose centre is $(3, -2)$ and radius $\sqrt{29}$. This square root should be found by geometry rather than by arithmetic. This is easily done since $29 = 5^2 + 2^2$.

Equation ii is $(x+2)(y-3) = 0.$

The graph consists of the two straight lines whose equations are $x = -2$, $y = 3$ respectively.

163. Maximum and minimum values.

By means of graphs, maximum and minimum values can

be approximately found which can be found accurately only with the aid of the Differential Calculus.

Example I. *Find the minimum value of* $3x^2-2x-4$.

The graph of $y = 3x^2-2x-4$ is the parabola shown in Fig. 41.

It is seen that the value of y is least at the turning point V. At the point V, $x = \frac{1}{3}$, $y = -4\cdot3$.

Hence $3x^2-2x-4$ is least when $x = \frac{1}{3}$, and its value is then $-4\cdot3$.

With a larger scale, these values can be found more accurately.

Another method. Quadratic functions can be dealt with as follows:

$$\begin{aligned} 3x^2-2x-4 &= 3\{x^2-\tfrac{2}{3}x\ \ \}-4 \\ &= 3\{x^2-\tfrac{2}{3}x+\tfrac{1}{9}\}-4-\tfrac{1}{3} \\ &= 3\,(x-\tfrac{1}{3})^2-4\tfrac{1}{3}\,; \end{aligned}$$

but a square quantity cannot be less than nothing. Hence the right-hand side is least when $(x-\frac{1}{3})^2 = 0$, and the required minimum value is $-4\frac{1}{3}$.

Example II. *Find the maximum and minimum values of* x^3-2x^2-x+2.

Fig. 40 shows that this function continually increases as x increases from $-\infty$ until x is $-\frac{1}{2}$ approximately, the value then begins to diminish. Hence, when $x = -\frac{1}{2}$, the function is a maximum, and its value is then 2·1.

As x increases from $-\frac{1}{2}$ to 1·5 the function continually decreases and afterwards increases. Hence, when $x = 1\cdot5$, the function has a minimum value, its value then being $-\cdot6$.

Notice that maximum and minimum values are not necessarily the same as greatest and least values. In this case the greatest and least values are respectively $+\infty$ and $-\infty$.

A maximum value of a function of x is a value which is greater than the values obtained when x is either slightly diminished or slightly increased; and a minimum value is one which is less than the values obtained when x is either slightly diminished or slightly increased.

Examples XVIII d.

Solve graphically:

1. $y = -0{\cdot}4x^2 + 1{\cdot}2x - 0{\cdot}8$, $y = 0{\cdot}5x - 1{\cdot}1$.

2. $y = {\cdot}7x^2 - 2{\cdot}2x + 1{\cdot}8$, $y = {\cdot}3x - 4$.

3. $x^2 - 2x = \frac{x^2}{4} + \frac{1}{2}$.

4. $y = 2x^2 - 6x + 3$, $y = x^2 - 2x - 1$.

5. What value of y will give equal roots for the equation
$$x^2 - 3x + 4 = y?$$

6. $x^2 - 10{\cdot}1x^2 + 25{\cdot}4x = 0$.

7. $y = x + 2 + \frac{1}{x}$, $y = x^2 + 2x + 1$.

8. $x^3 + 2x^2 - x - 1 = 0$.

9. $x^4 - 4x = 8$. Verify by drawing graphs of $y = x^2$, $y^2 - 4x = 8$.

10. $x^3 + 4{\cdot}84x + 2 = 0$.

11. Find graphically the maximum and minimum values of
$$\frac{x^2 + 6}{2x + 1}.$$

12. Find the least value of $x^2 + x - 2$ (i) algebraically, (ii) graphically.

13. Find graphically and algebraically the maximum or minimum value of $3 + 6x - x^2$.

14. Find the maximum and minimum values of
$$x^3 - 2x^2 - 5x + 3.$$

15. Divide the number 20 into two parts, such that the square of one together with three times the square of the other shall be a minimum.

16. A number is added to 2·25 times its reciprocal; for what number is the sum a minimum?

17. Draw the graphs of $y = 3 - 2x$ and $y = \frac{3}{x + 3}$ for values of x from -2 to $+3$, and hence find the positive root of
$$2x^2 + 3x - 6 = 0.$$

164. Graphs may be used instead of tables for reduction purposes, or to find square roots, &c.

Example I. *Draw a graph by means of which English money may be reduced to French money; it being given that* £1 = 25·3 francs.

Let £x be equivalent to y francs,
then $25{\cdot}3x = y$.
The graph is therefore a straight line.
When $x = 0$, $y = 0$; when $x = 2$, $y = 50{\cdot}6$.

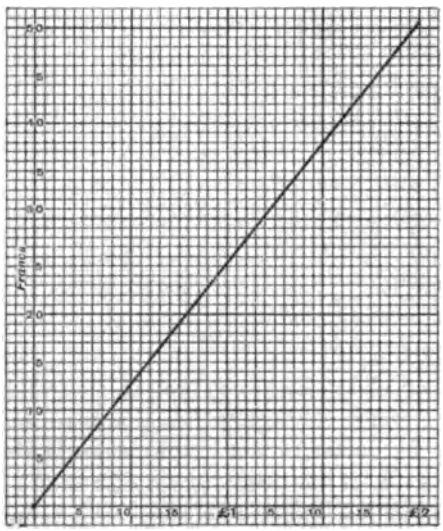

FIG. 42.

Fig. 42 shows the graph, and by means of it any sum less than £2 may be expressed in francs; and any sum less than 50 francs may be expressed in shillings.

Example II. *Draw a graph by means of which the cube roots of numbers from* 0 *to* 10 *may be found approximately.*

Here $y = \sqrt[3]{x}$, and graph must be drawn by plotting.
We have $x = y^3$.

y	1	2	3	1·5	2·5	2·25	·5	·8
x	1	8	27	3·4	15·6	11·4	·13	·5

The graph is shown on a small scale in Fig. 43; from it we see that $\sqrt[3]{2} = 1{\cdot}22$, $\sqrt[3]{4{\cdot}8} = 1{\cdot}68$, &c.

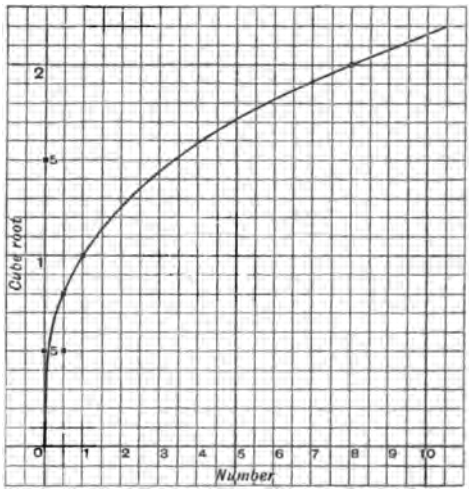

FIG. 43.

165. When observations are plotted, the resulting graph may suggest a law connecting the quantities under observation. If, for instance, the graph looks like a parabola, the law may be of the form $y = a + bx + cx^2$. By substituting the observed values, this suggested formula may be tested and possibly found to be true.

Example.

Simultaneous values of x *and* y *are given in the following table:*

x	0	1	2	3	4	5	6	7	8
y	3·25	3·45	3·65	5·0	6·45	8·25	10·45	13·0	16·0

Plot these values and endeavour to find a simple relation between x *and* y.

The graph looks like a parabola with its vertex at (0, 3·25).
Assume therefore as equation

$$y = 3{\cdot}25 + ax^2.$$

When $x = 1$, $y = 3{\cdot}45$; $\therefore\ 3{\cdot}45 = 3{\cdot}25 + a$,

i.e. $a = {\cdot}2$.

We have, therefore, as the suggested relation

$$y = 3{\cdot}25 + {\cdot}2x^2.$$

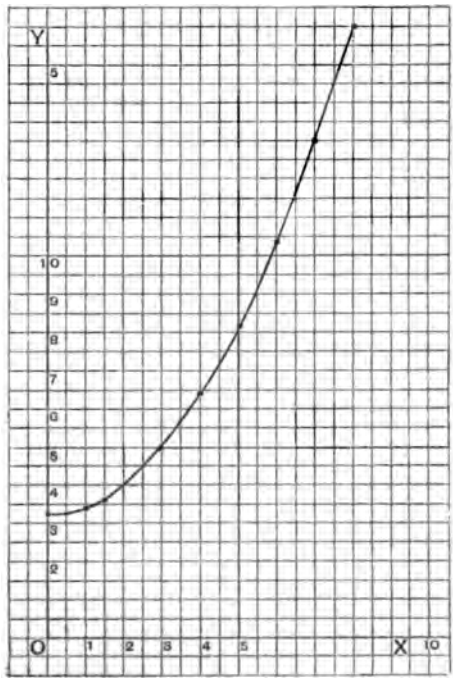

FIG. 44.

Test this by substituting the other values for x.

When $x = 3$, $3{\cdot}25 + {\cdot}2x^2 = 5{\cdot}05$.
When $x = 4$, $3{\cdot}25 + {\cdot}2x^2 = 6{\cdot}45$.
When $x = 5$, $3{\cdot}25 + {\cdot}2x^2 = 8{\cdot}25$.
When $x = 6$, $3{\cdot}25 + {\cdot}2x^2 = 10{\cdot}45$.
When $x = 7$, $3{\cdot}25 + {\cdot}2x^2 = 13{\cdot}05$.
When $x = 8$, $3{\cdot}25 + {\cdot}2x^2 = 16{\cdot}05$.

Examples XVIII e.

1. Draw a graph by means of which the square roots of numbers from 0 to 5 may be found to two decimal places. Use it to find $\sqrt{3{\cdot}6}$, $(1{\cdot}71)^2$.

2. Make a graph to reduce inches to centimetres; given 39·4 centimetres = 36 inches.

3. Freezing-point on a Fahrenheit thermometer is 32°, on a Centigrade thermometer 0°; the boiling points are 212°, 100° respectively. Draw a graph by means of which any Centigrade reading between 0 and 100 may be expressed in the Fahrenheit scale?

What is the equation of the graph?

4. Plot the following points:

x	-3	-1	3	4
y	-7	-1	11	14

If the points lie on a straight line find its equation.

5. When a weight W is tied to the end of a vertical elastic cord, the following observations of the length l are made:

W	300	500	1000	1500	1700
l	98·7	100·2	104·4	108·3	110·4

It is known that the graph obtained by plotting the values ought to be a straight line; find (i) the probable length of the cord when no weight is attached, (ii) which of the five observations is most inaccurate.

6. Two quantities, U and C, are found to have the following simultaneous values:

U	0·67	1·00	1·366	1·46	2·49
C	4·84	6·25	8·60	9·11	14·25

Is there any simple approximately correct law connecting U and C?

7. Draw a graph to show the simple interest on £10 at 3½ per cent. for any number of days less than 101 days.

Hence find the interest on £15 for 39 days at 3½ per cent.

8. The following simultaneous values of two quantities, c and s, are known. Find a simple relation connecting c and s.

c	1	·98	·94	·77	·71	·57	·42	·10
s	0	·17	·34	·64	·71	·82	·91	·99

9. Find a simple relation connecting x and y as given by the following table:

x	2	3	6	9	14	17
y	5	3·33	1·67	1·11	·71	·59

10. The time t secs. that a pendulum of length l feet takes to oscillate is given in the following table:

l	6	5	4	3	2	1
t	2·7	2·46	2·2	1·9	1·56	1·1

Show, by plotting the values of l and t^2, that a relation of the form $t = \sqrt{cl}$ exists, and find c from the graph.

Examples XVIII.

1. Draw graphs to represent the two expressions $4x+5$ and x^2; and thence obtain the solutions of the equation $x^2 = 4x+5$.

2. Plot the graph $y = x^2-8x+15$ between the values $x = -1$ and $x = 6$, thus obtaining the minimum value of y.

3. Plot the following values of x and y, and hence find the values of x when $y = ·89$ and of y when $x = ·8$.

x	0	·3	·7	·9	1·3	1·6	2	2·4	2·6	3
y	0	·29	·64	·78	·96	1	·91	·68	·52	·14

4. Solve by a graphical method $x^2+3x = 2$.

5. Solve by a graphical method

$$y-5 = 2(x-7),$$
$$y^2 = 21+4x-x^2.$$

6. A volume of gas occupies a volume v when the pressure is p; the following sets of simultaneous values of p and v are observed. Determine whether any simple relation connects p and v; and if so find it.

p	50	76	110	135	150	200	240
v	15·2	10	6·9	5·6	5·1	3·8	3·1

7. Plot the graph of $(y-1)^2 = 8(x+2)$.

8. Draw the graphs of $y = \frac{x}{3} + \frac{20}{3x}$ and $y = 3x-9$, and hence solve the equations $x^2-3xy+20 = 0$, $y-3x+9 = 0$.

9. In an examination the top boy obtains 215, the bottom boy 96; it is required to change the marks so that the top boy may

receive 90 and the bottom boy 25. Draw a graph by means of which the new marks of the intermediate boys may be found. [First show algebraically that the graph is a straight line.]

10. Draw a graph by which the cube root of any number between 0 and 10 may be found. By means of it find

$$\sqrt[3]{4{\cdot}2},\quad \sqrt[3]{7},\quad \sqrt[3]{9{\cdot}6}.$$

11. Draw the graph of $y = \dfrac{(1+2x)(2+x)}{x}$, and hence show that the right-hand side can never lie between 1 and 9.

12. Solve $x+y=7$, $xy=11$, and verify by drawing the graphs of the equations.

13. Draw the graph passing through the following points; find the equation of the graph:

x	0	·1	·3	·8	1	1·7
y	0	5·72	17·2	45·8	57·3	97·4

14. Draw the graph of $y = x + \dfrac{1}{x+2} + 3$ for values of x between -6 and 2.

15. Solve graphically $x+y=xy$ and $8x+7y=3xy$.

16. When $100y = x^3 - 100x$, calculate the values of y for $x = 8{\cdot}5$, 9, 10, 11, and draw the graph from $x = 8{\cdot}5$ to $x = 11$, showing the scales clearly.

17. Solve graphically $9x^2+4y^2=180$, $xy+12=0$. Verify by an algebraical solution.

18. Show graphically that the values of x and y which satisfy the simultaneous equations $5x = 2y-18$, $5y = 6-7x$, satisfy also the equation $x+y=2$.

19. Solve graphically $2(x-1)(y-1) = 6(x+y) = -3xy$.

20. The following table shows how the volume of a constant weight of water varies at different temperatures. Plot a graph and state what important information can be gathered from the graph.

Temperature	1°	3°	5°	7°	9°
Volume	27·6	25·3	25·2	25·8	27·3

21. By giving a few values to x, show on squared paper the part of the curve $y^2 - 3xy + (1+y)x^2 = 0$ which lies between $x=0$ and $x=1$.

22. A certain function of x is of the form $a+bx+cx^2$, where

a, b, c are constants, such that when x has the values -1, 1, 3 the values of the function are -2, 8, 10 respectively. Determine the values of a, b, and c, and plot a graph of the function between $x = -1$ and $+6$.

23. The distances of an object and its image from a lens, focal length 8 inches, are connected by the relation $\frac{1}{v} - \frac{1}{u} = \frac{1}{8}$, where v inches is the distance of the image and u inches of the object. Draw the graph of this equation, and hence find the value of u when the two distances are equal, and the value of v when $u = \infty$.

24. The following table shows simultaneous values of x and $f(x)$. Draw a graph, and hence find approximately for what values of x $f(x) = 2$.

x	0	·31	·58	·79	·91	1·07	1·24	1·36	1·43
$f(x)$	0	·33	·65	1	1·3	1·80	2·90	4·71	7·12

Graphical Problems.

25. A man starts from a place A to walk at the rate of 4 miles an hour. Draw a graph to show his distance from A at the end of every quarter of an hour. Draw a graph to show that at the end of 10 miles he rested for half an hour and then walked back to A at the rate of 3 miles an hour. From your graph determine the whole time of his walk.

26. Two men start walking towards one another at noon from places 12 miles apart. If one walks at 4 miles an hour, and the other at $3\frac{1}{2}$ miles an hour, determine by a graphical construction when and where they meet.

27. If, in question 14, the faster walker walks half-way and then sits down to wait for the other, find by a graph how long he will have to wait.

28. A sets out from a certain place and travels at the rate of 7 miles in 5 hours; 8 hours afterwards B sets out from the same place and travels the same road at the rate of 5 miles in 3 hours. How long will it take B to overtake A?

29. A and B start at the same time from two places 154 miles apart. A cycles at the rate of 11 miles an hour, B is in a motor travelling at the rate of 23 miles an hour. If they started at noon, when will they meet?

30. A point moves along a straight line so that its distance from one end O at any time x seconds is $x+\frac{10}{x}$ centimetres. Draw a graph showing the distance from O at any time, and find at what time the point is nearest O.

31. If A can make a box in 5 hours, and B a similar box in 4 hours, find by a graphical method how long it will take them to make 6 boxes working together.

32. A tap A can fill a cistern in 4 hours; a tap B can empty it in 7 hours. If both are turned on together how long will it be before the cistern is full?

33. A starts at 3 o'clock walking 4 miles an hour; B follows in the same direction at 3.15 at $4\frac{1}{2}$ miles an hour; then C rides in pursuit, starting at 4 o'clock, at 7 miles an hour. When and where does B catch A, C catch A, C catch B?

CHAPTER XIX

SIMULTANEOUS QUADRATICS

166. If either of two simultaneous equations is of the second degree, they are called simultaneous quadratics. Just as in the case of solving simple simultaneous equations, we try to solve by eliminating one of the unknown letters. This cannot always be done, and sometimes, when it is done, the resulting equation is of a higher degree than the second, and cannot be solved by elementary methods. No method applicable in all cases can be given, but there are two methods that can be used so often that they should be learnt thoroughly.

167. Method I. One of the equations is linear, i. e. of the first degree.

Example I. *Solve* $3x-2y=13$, (i)

$2x^2+3y^2=30$. (ii)

From i express one of the letters in terms of the other.

$$3x = 2y + 13.$$

$$\therefore \quad x = \frac{2y+13}{3}. \qquad \text{(iii)}$$

Substitute this value in ii.

$$2\left(\frac{2y+13}{3}\right)^2 + 3y^2 = 30.$$

Simplify $\quad 8y^2 + 104y + 338 + 27y^2 = 270,$

$$35y^2 + 104y + 68 = 0.$$

Solve $\quad (35y + 34)(y + 2) = 0.$

$$\therefore \quad y = -2 \quad \text{or} \quad y = -\tfrac{34}{35}.$$

Substitute in iii.

Then $\quad x = \dfrac{-4+13}{3} \quad \text{or} \quad x = \dfrac{-\frac{68}{35}+13}{3}$

$$= 3 \qquad\qquad = \frac{455-68}{3\times 35}$$

$$= \frac{387}{3\times 35}$$

$$= \frac{129}{35}.$$

Solution $\quad x = 3, \quad y = -2;$

or $\quad x = 3\tfrac{24}{35}, \quad y = -\tfrac{34}{35}.$

The answer should not be given

$$x = 3 \text{ or } 3\tfrac{24}{35},\ y = -2 \text{ or } -\tfrac{34}{35},$$

but the corresponding values of x and y should be given together.

This method can be used whenever one of the equations easily gives one unknown in terms of the other.

Example II. *Solve* $3x^2 - 4y^2 = -88,$ (i)

$$2xy = -20, \qquad \text{(ii)}$$

From ii $\quad 2y = -\dfrac{20}{x}. \qquad \text{(iii)}$

Substitute in i $\quad 3x^2 - \dfrac{400}{x^2} = -64.$

Simplify $3x^4+88x^2-400=0,$

$$(3x^2+100)(x^2-4)=0.$$

$$\therefore\quad x^2=4 \quad \text{or} \quad -\frac{100}{3}.$$

Extract square root

$$x=+2 \text{ or } -2 \text{ or } +10\sqrt{-\tfrac{1}{3}} \text{ or } -10\sqrt{-\tfrac{1}{3}}.$$

Substitute in iii

$$y=-5 \text{ or } +5 \text{ or } -\frac{1}{\sqrt{-\frac{1}{3}}} \text{ or } +\frac{1}{\sqrt{-\frac{1}{3}}}.$$

Hence the real solutions are $x=2,\ y=-5;$

or $x=-2,\ y=+5.$

Example III. *Solve* $3x^2-2x+y^2=37,$ (i)

$$xy=-12. \quad \text{(ii)}$$

From ii $$y=-\frac{12}{x}. \quad \text{(iii)}$$

Substitute in i $$3x^2-2x+\frac{144}{x^2}=37.$$

Simplify $$3x^4-2x^3-37x^2+144=0.$$

This is of the 4th degree and contains x^2; it cannot be solved by treating it as a quadratic with x^2 as the unknown, as we did in Example II. Our only chance of solving is by using the factor theorem, i. e. we try to find a root by trial.

The possible roots are the factors of 144, viz. 1, 2, 3, 6, 8, 12, &c.

$f(1)=147-39,\ f(-1)=149-37,\ f(2)=48-16-148+144,$

$f(-2)=48+16-148+144,\ f(3)=243-54-333+144=0.$

$\therefore\ x=3$ is one root and $x-3$ is a factor of the left-hand side.

Hence one solution is $x=3,\ y=-4.$

Divide by $x-3,$

$$3x^2(x-3)+7x^2(x-3)-16(x^2-9)=0.$$

$$\therefore\ 3x^3+7x^2-16x-48=0.$$

By the factor theorem or by the graph method at least one root of this equation can be found; by substitution in iii the corresponding value of y is found.

The factor method can find only rational roots, the graph method will give approximate values of all real roots.

Examples XIX a.

Solve:

1. $2x+3y=8$, $xy=2$.
2. $3x-4y=1$, $3x^2-4y^2=11$.
3. $y-2x=7$, $y^2-4x^2=105$.
4. $3x+y=1$, $9x^2+y^2=61$.
5. $6xy=1$, $(x+1)(y+1)=2$.
6. $x-2y=3$, $x^2-2xy+3y^2=33$.
7. $x^2-xy+y^2=+19$, $x+y=1$.
8. $x^2-xy+y^2=4$, $x+y=1$.
9. $x+ay=1$, $x^2+2ay^2=a^2+1$.
10. $3x-5y=1$, $x^2-2xy-3y^2=-15$.
11. $\dfrac{10x+y}{xy}=3$, $9y-9x=18$.
12. $\dfrac{x+y}{x-y}=\dfrac{13}{5}$, $y^2+x=25$.
13. $x^3+y^3=10$, $x+4y=7$.
14. $5xy+3x-y=492$, $\dfrac{y}{2x}=\dfrac{3}{4}$.
15. $x^3-xy=153$, $x+y=1$.

16. There are two numbers whose difference is $\frac{2}{3}$ of the greater, and the difference of their squares is 128, find them.

17. The difference of two numbers is 15, and half their product equals the cube of the smaller number, find them.

18. The product of two numbers is 56; their difference is 10; find the sum of their squares.

19. A regiment consisting of 1066 men is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each square?

20. A poulterer bought 15 ducks and 12 turkeys for 5 guineas. He had 2 ducks more for 18 shillings than he had turkeys for £1. What was the price of each?

168. Method II. Homogeneous Equations.

When every term in the two equations that contains the unknowns is of the same degree, the equations are said to

be homogeneous. In such cases **Eliminate the constant term and factorise the left-hand side of the resulting equation.**

Example. *Solve* $x^2-3xy-y^2=9$, (i)

$2x^2+2xy+3y^2=7$. (ii)

Multiply i by 7 $7x^2-21xy-7y^2=63$.

Multiply ii by 9 $18x^2+18xy+27y^2=63$.

Subtract $11x^2+39xy+34y^2=0$.

Resolve into factors $(11x+17y)(x+2y)=0$;

either $11x+17y=0$ or $x+2y=0$,

i.e. $x=-\frac{17}{11}y$ or $x=-2y$.

Substitute in equation i

$\frac{289}{121}y^2+\frac{17}{11}y^2-y^2=9$ or $4y^2+6y^2-y^2=9$,

$(289+561-121)y^2=9\times121$, $9y^2=9$,

$729y^2=9\times121$, $y^2=1$;

$\therefore\ y=+\frac{11}{9}$ or $-\frac{11}{9}$, $\therefore\ y=+1$ or -1,

$x=-\frac{17}{9}$ or $+\frac{17}{9}$; $x=-2$ or $+2$,

each value of x being taken with the value of y above it.

Examples XIX b.

Solve:

1. $3x^2+4y^2=16$, $4xy=8$.
2. $x^2+xy=15$, $xy-y^2=2$.
3. $x^2-2y^2=17$, $x^2+2y^2=33$.
4. $x^2-2xy-y^2=7$, $2x^2-3xy+4y^2=62$.
5. $3x^2-7y^2=5$, $3xy-5y^2=1$.
6. $x^2+xy+y^2=52$, $xy-x^2=8$.
7. $4x^2+9y^2=10$, $2xy+1=0$.
8. $4x^2-xy=14$, $3xy-y^2=5$.
9. $x^2-2xy-y^2=7$, $xy+2y^2=12$.
10. $\frac{x}{y}+\frac{y}{x}=\frac{61}{30}$, $x^2+xy+y^2=91$.
11. Find two numbers such that the sum of their squares is 29 and their product 10.

12. The product of two numbers is 24, and their sum multiplied by their difference is 20, find them.

13. Find two positive integers whose sum multiplied by the greater is 204 and whose difference multiplied by the less is 35.

14. Divide 14 into two integral parts so that the sum of the quotients obtained by dividing each part by the other may be $2\frac{1}{12}$.

15. A man buys tea at two different shops spending altogether £2. 17*s.* 1*d.*; the price in pence per lb. at each shop being equal to the number of pounds bought at that shop. If the price at each shop had equalled as many pence as he bought pounds at the other shop, he would have spent 1*d.* less. How many pounds did he buy at each shop?

169. Miscellaneous Methods.

There are various devices which are useful in solving simultaneous quadratic equations; some of these are shown in the following examples.

Example I. *Solve* $3x+4y = 18$, (i)

$xy = 6$. (ii)

Square i $9x^2+24xy+16y^2 = 324$.

Multiply ii by 48 $48xy = 288$.

Subtract $9x^2-24xy+16y^2 = 36$.

Extract square root $3x-4y = +6$ or -6.

Add to equation i $6x = 24$ or 12.

$\therefore\ x = 4$ or 2.

From ii if $x = 4$, $y = 1\frac{1}{2}$;

if $x = 2$, $y = 3$.

Example II. $x^2-xy = 84$, (i)

$xy-y^2 = 48$. (ii)

Subtract $x^2-2xy+y^2 = 36$.

Take square root $x-y = +6$ or -6. (iii)

Divide equation i by iii $x = 14$ or -14.

Divide equation ii by iii $y = 8$ or -8.

Solution $x = 14$, $y = 8$;

$x = -14$, $y = -8$.

Example III. *Solve* $2x+5xy = 34,$ (i)

$3y+4xy = 33.$ (ii)

Eliminate xy.

Multiply i by 4 $\quad 8x+20xy = 136.$

Multiply ii by 5 $\quad 15y+20xy = 165.$

Subtract $\quad 8x-15y = -29.$

And solution may be finished by Method I.

Example IV. *Solve* $x^2y^2-5xy+6 = 0,$ (i)

$x^2+2y^2 = 11.$ (ii)

Treat equation i as a quadratic to find xy.

Resolve i into factors $(xy-3)(xy-2) = 0,$

$xy = 2$ or $3.$

And the solution may be finished by Method I or II.

16.V.22

170. Graphical Method.

Draw the graph of each equation; then the co-ordinates of the points of intersection satisfy both equations, i. e. the co-ordinates of the points of intersection give the required solution (see also § 162).

Example. *Solve* $y = x^2-3x+2,$ (i)

$x = 2y^2-3.$ (ii)

Equation i is not of the first degree, ∴ the graph is not a straight line and at least six points must be found.

x	-3	-2	-1	0	1	2	3	4	5
x^2	9	4	1	0	1	4	9	16	25
$-3x$	9	6	3	0	-3	-6	-9	-12	-15
y	20	12	6	2	0	0	2	6	12

Equation ii is not of the first degree, ∴ the graph is not a straight line.

y	-3	-2	-1	0	1	2	3	4
$2y^2$	18	8	2	0	2	8	18	32
x	15	5	-1	-3	-1	5	15	29

In Fig. 45 the points of intersection are P and Q, and the solution is approximately $x = \cdot3$, $y = 1\cdot3$ or $x = 2\cdot8$, $y = 1\cdot6$.

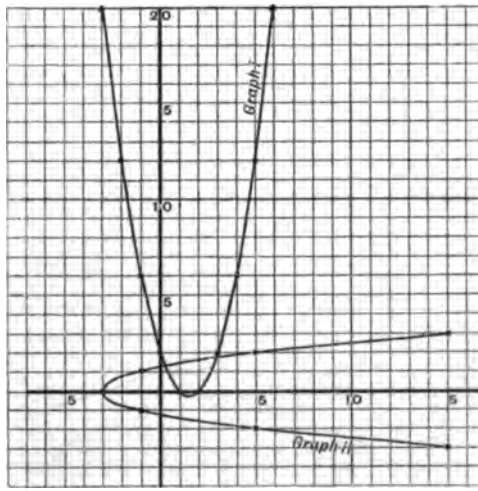

FIG. 45.

To obtain a more accurate solution that portion of the graphs lying between $x = 0$, $x = 3$ should be drawn on a larger scale.

Graph i

x	·1	·2	·3	·4	·5	·6		2·3	2·5	2·6	2·7	2·8	2·9	3
$2+x^2$	2·01	2·04	2·09	2·16	2·25	2·36		7·29	8·25	8·76	9·29	9·84	10·41	11
$3x$	·3	·6	·9	1·2	1·5	1·8		6·9	7·5	7·8	8·1	8·4	8·7	9
y	1·71	1·44	1·19	·96	·75	·56		·39	·75	·96	1·19	1·44	1·71	2

Graph ii

y	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8
y^2	1·21	1·44	1·69	1·96	2·25	2·56	2·89	3·24
$2y^2$	2·42	2·88	3·38	3·92	4·50	5·12	5·78	6·48
x	−·58	−·12	·38	·92	1·50	2·12	2·78	3·48

These values are shown plotted in Fig. 46 and the co-ordinates of the point of intersection are there seen to be

$$x = \cdot 26, \quad y = 1 \cdot 28;$$
$$x = 2 \cdot 91, \quad y = 1 \cdot 71.$$

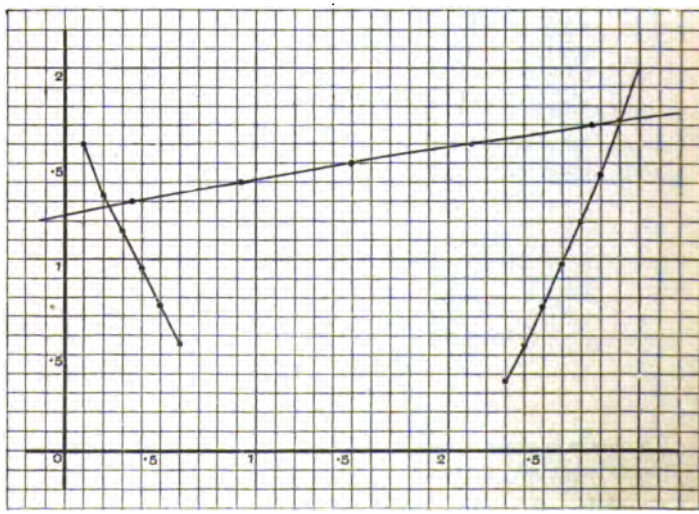

Fig. 46.

On substituting in the original equations it will be found that these are very close approximations to the exact solution.

Examples XIX.

Solve:

1. $x^2 + y^2 = 169,$
$xy = 60.$

2. $9x^2 - 16y^2 = 65,$
$3x - 4y = 13.$

3. $x^3+y^3=35$, $x^2y+xy^2=30$.

4. $x^2-y^2=9$, $2x^2+3y^2=98$.

5. $x^2y^2+16=8xy$, $3x-4y=8$.

6. $xy=20-x^2=y^2-6$.

7. $x^2=y+12$, $y^2=x+12$.

8. $x^2+x+y=18-y^2$, $xy=6$.

9. $x^2+2xy+y^2+2x=120-2y$, $xy-y^2=8$.

10. $x^2+y^2-x-y=78$, $xy+x+y=39$.

11. $\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9}$, $x-y=2$.

12. $x^3+y^3+x-y=132$, $(x^3+y^3)(x-y)=1220$.

13. $(x-y)^2=8(x+y)$, $(x+y)^2=8(x-y)$.

14. $y^3-3x^2+y+3x+30=0$, $y^3+x^2+y-x-18=0$.

15. $4x^2+5xy=6$, $2y^2-3xy=26$.

16. $\frac{x}{y}+\frac{2y}{x}=3$, $(x+y)(x+2y)=12$.

Solve graphically:

17. $y^2-7x-8y+14=0$, $7x+6y=13$.

18. $x-y^2=3y$, $y-3x=0$.

19. $x=3y^2-2y+1$, $3y-4x+5=0$.

20. $x=3y-y^2$, $y^2=4x-2$.

CHAPTER XX

MISCELLANEOUS PROBLEMS

171. Many problems have been set in the previous Example; before giving a set of miscellaneous problems we will work a few to show the kind of way the necessary equations are arrived at. The student is referred to § 109, where some hints have already been given.

Example I. *A merchant gained as many shillings on a certain number of pounds of tea as there were shillings in the cost price of* $10\frac{1}{2}$ *lb. or as there were farthings in the selling price of a* 8 *oz. packet. How many pounds did he sell?*

Let x shillings be selling price of 1 lb., and y shillings be cost price of 1 lb.

Then $x-y$ shillings is gain on 1 lb.

Also $\frac{21}{2}y$ shillings is the cost price of $10\frac{1}{2}$ lb., and $\frac{3}{16}x \times 48$ farthings is the selling price of 3 oz.

$\therefore$ his total gain according to the question is either $\frac{21}{2}y$ shillings or $9x$ shillings. Hence an equation is

$$9x = \tfrac{21}{2}y,$$

$$\text{i.e.} \quad 6x = 7y. \tag{i}$$

Again total gain is $9x$ shillings; gain on one pound is $x-y$ shillings.

$$\therefore \text{ number of pounds} = \frac{9x}{x-y}$$

$$= \frac{9x}{x-\frac{6}{7}x} = 63. \quad \textit{Answer.}$$

This is an example where we do not assume the unknown to be the quantity we wish to find.

Example II. *Two sums of money, making together £63, are spent on two different kinds of articles whose prices differ by* 1*s., the number of articles obtained being* 190. *If the two sums are interchanged the number of articles obtained will be* 200. *Find the prices and the sums of money.*

Let x shillings be sum of money spent on articles at y shillings each.

Then $(1260-x)$ shillings is spent on articles at $(y-1)$ shillings each.

$\therefore$ $\frac{x}{y}$ is number of first kind; and $\frac{1260-x}{y-1}$ is number of second kind.

Hence from the question,

$$\frac{x}{y} + \frac{1260-x}{y-1} = 190. \tag{i}$$

If x shillings had been spent on articles at $(y-1)$ shillings each and $(1260-x)$ shillings had been spent on articles at y shillings each, the numbers bought would have been $\frac{x}{y-1}$ and $\frac{1260-x}{y}$ respectively.

Hence $$\frac{x}{y-1}+\frac{1260-x}{y}=200. \qquad \text{(ii)}$$

We now have two equations to find two unknowns.

Add equations i and ii $$\frac{1260}{y}+\frac{1260}{y-1}=390.$$

Multiply by $y(y-1)$ and divide by 30,

$$42(2y-1)=13(y^2-y).$$

Transpose $$13y^2-97y+42=0.$$

Factorize $$(13y-6)(y-7)=0.$$

$$\therefore\ y=7 \quad \text{or} \quad \tfrac{6}{13}.$$

Hence the solution of the equation gives two alternatives, viz. 7*s.* and 6*s.* as the respective prices of an article, or $\frac{6}{13}$ and $-\frac{7}{13}$ shillings as the prices.

A negative price is absurd, therefore the solution $y=\frac{6}{13}$ is rejected.

If $y=7$, we have, by substituting in equation i,

$$\frac{x}{7}+\frac{1260-x}{6}=190.$$

Multiply by 42 $$6x+7\times1260-7x=42\times190.$$

Transpose $$-x=42\times190-42\times210$$

$$=-42\times20.$$

$\therefore$ one sum of money is 42×20 shillings = £42, which is spent on articles at 7*s.* each;

and the other sum of money is £21, which is spent on articles at 6*s.* each.

Note. If $y=\frac{6}{13}$ had been found to be a reasonable solution, the price would have been given not as $\frac{6}{13}$*s.* each, but as 6 shillings for 13.

Example III. *The difference in the rates of two trains is* 18 *miles per hour. The faster of the two takes* 2 *hours less time to travel* 164 *miles than the slower takes to travel* 168 *miles. Find their respective rates.*

Let x miles per hour be the rate of the faster.

Then $\frac{164}{x}$ hours is the time it takes to travel 164 miles.

Also $x-18$ miles per hour is the rate of the slower train.

Then $\frac{168}{x-18}$ hours is the time it takes to travel 168 miles.

From the question this second time is 2 hours more than the first time.

$$\therefore \quad \frac{168}{x-13} - \frac{164}{x} = 2.$$

Multiply by $x(x-13)$,

$$168x - 164x + 164 \times 13 = 2x^2 - 26x.$$

Transpose and divide by 2,

$$x^2 - 15x - 82 \times 13 = 0.$$

Factorize $$(x-41)(x+26) = 0.$$

$$\therefore \quad x = 41 \quad \text{or} \quad -26.$$

The negative value is inconsistent with the meaning of the question. The only meaning we can attach to a rate of -26 miles per hour is that the train is travelling in the opposite direction to some train which is moving with positive velocity; that interpretation does not apply here.

∴ the fast train travels at 41 miles per hour,
the slow at 28 miles per hour.

Example IV. *What is the price of gold per ounce if a rise of 8d. reduces by 5 the number of ounces that can be bought for £5757?*

Let £x be original price per ounce.

Then £$(x + \frac{1}{80})$ is the increased price.

$\frac{5757}{x}$ is the original number of ounces for £5757.

$\frac{5757}{x+\frac{1}{80}}$ is the second number of ounces for £5757.

From the question

$$\frac{5757}{x} - \frac{5757}{x+\frac{1}{80}} = 5.$$

Multiply by $x(x+\frac{1}{80})$,

$$5757 \times \tfrac{1}{80} = 5x^2 + \tfrac{1}{16}x.$$

Multiply by 80 and transpose,

$$400x^2 + 5x - 5757 = 0,$$

$$(80x - 303)(5x + 19) = 0;$$

$$\therefore \quad x = \tfrac{303}{80} \quad \text{or} \quad -\tfrac{19}{5}.$$

The negative value is inconsistent with the question.

∴ required price is £3. 15*s*. 9*d*. per ounce.

172. Algebra can often be used to solve Geometrical questions, e.g.

Example. *Divide a straight line* 9 *inches long so that the square on one part may be double the square on the other.*

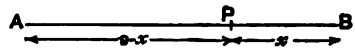

FIG. 47.

Let x inches be the length of the smaller part.
Then $9-x$ inches is length of the larger part.
Then $(9-x)^2 = 2x^2$.
Remove brackets and transpose,

$$x^2 + 18x - 81 = 0.$$

Solve by the formula $x = \dfrac{-18 \pm \sqrt{324+324}}{2}$

$$= \frac{-18+25{\cdot}4}{2} \text{ or } \frac{-18-25{\cdot}4}{2}$$

$$= \frac{7{\cdot}4}{2} \text{ or } -\frac{43{\cdot}4}{2}$$

$$= 3{\cdot}7 \text{ or } -21{\cdot}7.$$

	648 | 25·4
45	248
504	2300

Hence one portion is 3·7 inches long and the other 5·3 inches.

In this case the negative value would mean that BP was to be measured to the right instead of to the left and would give the answer to the question.

A straight line is 9 *inches long; produce it so that the square on the whole line, when produced, shall equal twice the square on the part produced.*

173. Problem leading to an indeterminate equation. Sometimes the data of the question give only one equation although there are two unknowns. In such cases the equation has an infinite number of solutions (§ 60); if, however, the conditions of the question require that the unknowns should be positive integers, the solutions are limited in number, but there may be some difficulty in

finding them. Problems and equations of this kind are said to be indeterminate.

Example. *In firing at a target the score is* 5 *for a bull,* 4 *for an inner,* 3 *for a magpie,* 2 *for an outer, and* 0 *for a miss. In how many ways can a score of* 30 *be made with* 7 *shots, it being known that there are no magpies or misses?*

Let x be the number of bulls.
y be the number of inners.
z be the number of outers.

Then
$$5x+4y+2z=30, \quad \text{(i)}$$
$$x+y+z=7; \quad \text{(ii)}$$

so that there are only two equations for three unknowns; but x, y, and z must be positive integers.

Eliminate z by subtracting twice ii from i,

then
$$3x+2y=16. \quad \text{(iii)}$$

Divide by the coefficient of x or y.

Divide by 3
$$x+\tfrac{2}{3}y=5+\tfrac{1}{3}.$$

Transpose integers to the right,
$$\tfrac{2}{3}y-\tfrac{1}{3}=5-x=\text{an integer}.$$

Multiply by an integer k,

then
$$\frac{2ky-k}{3}=\text{an integer}.$$

Choose k so that when $2k$ is divided by 3 the remainder is 1; in this case take $k=2$,

then
$$\frac{4y-2}{3}=\text{an integer},$$

i.e.
$$y+\frac{y-2}{3}=\text{an integer};$$

$$\therefore \quad \frac{y-2}{3}=\text{an integer}=t \text{ (suppose)};$$

then
$$y=3t+2, \text{ where } t \text{ is any integer.}$$

Substitute in iii
$$3x=16-2\times 3t-4;$$
$$\therefore \quad x=4-2t.$$

Substitute in ii
$$z=7-(3t+2)-(4-2t)$$
$$=1-t.$$

Hence if
$$t=-1 \quad 0 \quad 1 \quad 2$$

then
$$x=6 \quad 4 \quad 2 \quad 0$$
$$y=-1 \quad 2 \quad 5 \quad 8$$
$$z=2 \quad 1 \quad 0 \quad -1$$

If t is negative, y is negative; if $t>1$, z is negative; and the only possible solutions are when $t = 0$, or 1.

Therefore there are two ways of making a total 30 under the given conditions.

Graphical Method. Draw the graph of equation iii and notice where it passes through corners of squares.

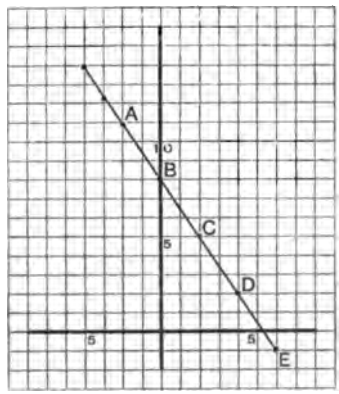

FIG. 48.

Since the equation is of the first degree, the graph is a straight line.

When $x = 0$, $y = 8$;

when $x = -4$, $y = 14$;

when $x = +6$, $y = -1$.

As shown in Fig. 48 the graph passes through the points (0, 8), (2, 5), (4, 2), and these give the only positive integral solutions of equation iii.

174. Questions asking for greatest or least values can often be solved by using the fact that a square quantity cannot be less than zero. They can also be solved by finding the turning-point of a graph.

Example. *What is the area of the biggest rectangular field that can be surrounded by* 400 *yards of fencing?*

Let x yards be the length of one side.

Then $(200-x)$ yards is the length of the other.

Therefore the area is $x(200-x)$ square yards;

but $$x(200-x) = 200x - x^2$$
$$= -(x^2 - 200x)$$
$$= -(x^2 - 200x + 100^2) + 100^2$$
$$= 100^2 - (x-100)^2;$$

$\therefore$ $x(200-x)$ is greatest when $(x-100)^2$ is least,

i.e. when $(x-100)^2 = 0$,

i.e. when $x = 100$.

Hence greatest area $= 100 \times 100$ square yards
$= 10000$ square yards.

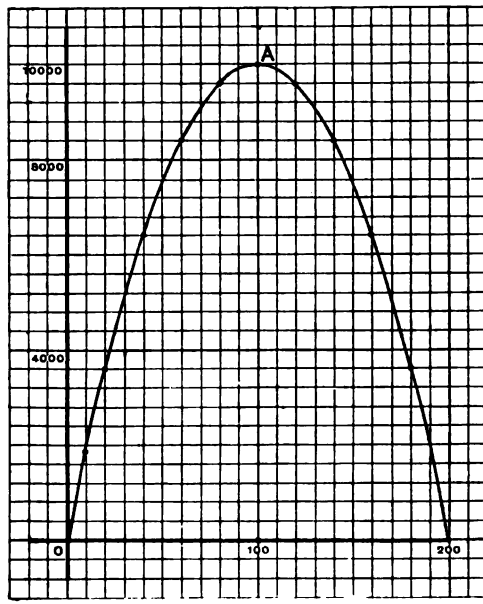

Fig. 49.

Graphical Method. Draw the graph of $y = x(200-x)$. This is shown in Fig. 49.

It is seen that the curve turns at the point A, and that the ordinate y is greater there than at any other point.

Hence the graph shows that y is greatest when $x = 100$, and that its greatest value is 10000.

The graphical method is specially useful when the expression whose maximum or minimum is required is higher than the second degree.

Examples XX.

1. One-fourth of the subscribers to a certain fund gave a sovereign apiece, one-fourth of the remainder gave half a sovereign apiece, and the rest gave a florin. If the three sets of subscribers raised their subscriptions to a guinea, half a guinea, and half a crown respectively, the total increase in the subscriptions would be £2. 10*s*. How many subscribers were there, and what was the total amount subscribed?

2. Two workmen take the same time to earn £22 and £21 respectively. The former earns £15. 8*s*. in one day less time than the latter takes to earn the same sum. How much does each earn per day?

3. A man walks 18 miles from *A* to *B* in 4 hours and 34 minutes, and does the return journey in 4 hours 50 minutes; walking 4 miles an hour on the level, 3 miles an hour uphill, and 5 miles an hour downhill. Find how many miles there are on the level, how many up, and how many down on his first journey?

4. A draper buys 150 yards of ribbon. Some of this is of fine quality, the rest is of an inferior quality and costs 1*s*. 6*d*. a yard less than the former. If he spends £6 on the dearer ribbon and £15 on the cheaper, what do the ribbons cost per yard respectively?

5. I bought a horse and dog for £25. If I had given 10 per cent. more for the dog and 2½ per cent. more for the horse, I should have spent another 20 shillings. What did I pay for each of them?

6. What is the length of a race in which *A* beats *B* by 5 seconds and *B* beats *C* by 1 second, the average pace of *A* being ¼ yard per second greater than that of *B*, and that of *B* ¼ yard per second greater than that of *C*?

7. A cyclist travels the first half of the distance from *P* to *Q* at the rate of 10 miles per hour; he then rests 15 minutes and travels the remainder of the distance at the rate of 15 miles per hour. Had he travelled the whole distance without stopping at the rate of 11 miles per hour he would have taken exactly the same time. Find the distance from *P* to *Q*. Verify by a graphical construction.

8. A certain sum of money is to be equally divided among a certain number of people. If the number of people had been greater by ten the share of each would have been smaller by 9*s.* 2*d.*; and if the number of people had been fewer by ten the share of each would have been larger by 13*s.* 4*d.* What was the sum of money to be divided?

9. A fruiterer sold a certain quantity of oranges for £6. 10*s.* If he had given two more oranges for a shilling the same quantity would only have realized £5. 17*s.* How many oranges did he sell?

10. A gravel path two yards wide is made round a square field, and it is found that it takes up one-twentieth of the area of the field. Prove that the area of the field is nearly 25000 square yards.

11. A person buys x articles for y pence and sells them at the rate of y articles for x pence, and he finds that his profit is 30 per cent. almost exactly. What is the value of $\frac{x}{y}$?

12. The perimeter of a rectangle is 340 feet and its diagonal is 130 feet; find the area.

13. The side of a square is 79 feet. Find at what points of the sides the vertices of an inscribed square must be placed in order that it may have an area of 4225 square feet.

14. The prices of seats in the stalls, pit, and gallery of a theatre are respectively seven shillings and sixpence, half a crown, and one shilling. The pit can hold three times and the gallery twice as many people as the stalls. The receipts are £87. 6*s.* when all the seats in the pit and gallery are occupied and all but twelve in the stalls; how many people are present?

15. Rowing at a uniform rate a person can row 5 miles down stream in 30 minutes and back again in 40 minutes; find the velocity of the stream.

16. By adding 30 yards to the length and 10 yards to the breadth of a square field its area is doubled; find its original area in square yards.

17. What number is such that when divided by the product of its digits the quotient is two, and when it is increased by 27 the digits are reversed?

18. A grocer sold 80 pounds of mace and 100 pounds of cloves for £65, but he sold 60 pounds more of cloves for £20 than he did of mace for £10. What was the price of each per pound?

19. The fore-wheel of a carriage makes 6 revolutions more than

the hind-wheel in going 120 yards; but if the circumference of each wheel be increased by one yard it will make only 4 revolutions more than the hind-wheel in the same distance. Find the circumference of each.

20. Find two numbers whose sum multiplied by the sum of their squares is 272, and whose difference multiplied by the difference of their squares is 32.

21. A certain train is 5 minutes late when it performs its journey at the rate of 29½ miles an hour, and it is two minutes late when it travels at 30 miles an hour; what is the length of the journey?

22. In a factory there are employed in one week 57 men, 14 women, and 10 boys, the total wages for the week being £84. 5*s*.; the next week 60 men, 17 women, and 11 boys are employed, the increase of wages amounting to £6. 5*s*. If two women receive as much as a man and a boy, find the weekly earnings of a man, woman, and boy respectively.

23. A person put out a certain sum to interest for 6½ years at 5 per cent. simple interest and found that if he had put out the same sum for 12 years and nine months at 4 per cent. he would have received £185 more. What was the sum invested?

24. An alloy of copper and tin containing 100 cubic inches weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh 5½ ounces and a cubic inch of tin to weigh 4½ ounces?

25. A number of two digits when multiplied by the left-hand digit gives product 46; if the sum of the digits be multiplied by the same digit the product is 10. What is the number?

26. A person being asked his age replied, 'If you add the square root of it to half of it and subtract 12 there will remain nothing.' Find his age.

27. *A* and *B* gained by trading £100, which is divided proportionately to the amount of capital each provided. Half of *A*'s capital was £100 less than *B*'s; and *A*'s share of the profits was $\frac{3}{10}$ of *B*'s capital. Find the capital of each.

28. There are two mixtures of whisky and water. *A* contains 30 per cent. of water, *B* 40 per cent. of water. If *A* is poured into *B*, the resulting mixture contains 36 per cent. of water. If 30 gallons of water are now poured in, the mixture contains 60 per cent. of water. Find the number of gallons in *A* and *B* respectively.

29. A boat which travels at the rate of 10½ miles an hour down

stream takes three times as long to go a certain distance up a river as to go the same distance down. Find the rate of flow of the stream.

30. A train runs a distance of 60 miles. If it had gone 10 miles an hour faster it would have taken half an hour less for the journey. Find the rate of the train.

31. The journey between two towns by one route consists of 233 miles by rail, followed by 126 miles by sea; by another route it consists of 405 miles by rail, followed by 39 miles by sea. If the time occupied on the journey is 50 minutes longer by the first route than by the second, find the average speed by rail, assuming it to be the same by each route and 25 miles an hour faster than the average speed by sea.

32. A and B travel from a town X to a town Y, a distance of $10\frac{1}{2}$ miles. They have a bicycle between them which they can ride at 7 and 10 miles an hour respectively, their walking speeds being $3\frac{1}{2}$ and 4 miles an hour respectively. They start together from X, B walking, A on the bicycle. On arriving at a point P, A leaves his bicycle and walks on. As he is walking into Y he is overtaken by B who has picked up the bicycle and ridden from P. Find the distances of P from X and Y.

33. A working 6 hours a day and B working 7 hours a day together do a piece of work in a certain time. If A worked 7 hours a day and B worked 10 hours a day they could do it in 2 days less. A is twice as rapid a worker as B. Find how many days the work takes.

34. Prove generally that with interest at x per cent. the commercial present worth is less than the true present worth by the fraction $\left(\frac{x}{100}\right)^2$ of the latter.

35. Find the lengths of the two parts into which a line 10 feet long must be divided in order that the areas of the squares on them may be together equal to 64 sq. feet. Give the answer correct to the hundredth of an inch.

36. The hypotenuse of a right-angled triangle is 91 feet long, and the longer of the two remaining sides exceeds the shorter by 49 feet; find the area.

37. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per cubic yard as there were yards in

a side of the other; and the base of the greater exceeded the base of the less by 9 square yards. What was the price of each?

38. The captain of a privateer seeing a trading vessel 7 miles ahead sailed 20 miles in direct pursuit of her; and then, observing that the trader was steering in a direction perpendicular to her former course, changed his own direction so that he overtook the trader without making any further change in his course. The rates of the privateer and trader were 10 miles and 7 miles an hour respectively. What distance did the privateer sail?

39. Divide a straight line 2 inches long into two parts so that the square on one part is double the square on the other part. Account for the two answers.

40. Divide a straight line 3 inches long so that the square on one part equals twice the rectangle contained by the whole and the other part.

41. A person buys two pieces of cloth for £15, the one at 8*s*., the other at 11*s*. per yard, each piece containing more than 10 yards; how many yards did he buy altogether?

42. In how many ways can £1 be paid in half-crowns, shillings, and sixpences, using exactly 18 coins?

43. There are a number of eggs in a basket; when counted 3 at a time there are 2 over; when counted 5 at a time there are 4 over. It is known that there are more than 50 but less than 100 eggs; how many are there?

44. How many pairs of fractions are there, one with denominator 12, the other with denominator 18, whose sum is $\frac{5}{6}$?

45. In a journey of 100 miles a man walked for an exact number of hours at 4 miles per hour and cycled for an exact number of hours at 10 miles per hour, how long did the journey take?

46. A stone is thrown into the air so that its distance from the ground after t secs. is $100t - 16t^2$, what is the greatest height it reaches? Verify by a graph.

47. A man receives 5 shillings a day while he is engaged on a certain piece of work, but his expenses are half as many shillings per day as he takes days to complete the work; what is the greatest sum he can earn, and how many days will he then take?

48. A cistern without a top and with a square base is to be made with 48 sq. feet of material. What is the maximum content of the cistern?

49. A person stands on the bank of a river which is 100 yards

wide; he wishes to get to a place A 300 yards along the stream on the opposite bank. He can row 3 miles an hour and run 5 miles an hour along the bank. Whereabouts should he land on the opposite bank so as to reach A as soon as possible?

50. The combined length and girth of a parcel sent by Parcel Post must not exceed 6 feet. Find the volume of the largest parcel that can be sent. (Assume that the parcel is cylindrical.) §106. $\frac{8}{\pi}$ cu. ft.

9.VI.22

CHAPTER XXI

INDICES

175. For convenience we repeat the definition of x^n.

x^n is the product of n factors each equal to x.

The coefficient 1 is understood, we have therefore adopted the following definition which has several advantages:

x^n is the result of multiplying unity by x n times.

We have also the definition of the n^{th} root, viz.:

$\sqrt[n]{x}$ is a quantity whose n^{th} power is equal to x, i.e.

$$\{\sqrt[n]{x}\}^n = x.$$

With these definitions we proceed to prove the following rules:

176. Multiplication Law.

If two powers of any quantity are multiplied together, the product is a power whose index is the sum of the original indices.

We have to show that $x^p \times x^q = x^{p+q}$.

By definition $x^p = 1 \times x \times x \times x \ldots p$ multiplications.

$x^q = 1 \times x \times x \times x \ldots q$ multiplications.

$\therefore \quad x^p \times x^q = 1 \times x \times x \times x \ldots$ (p multiplications) $\times x \times x \times x \ldots$ (q multiplications)

$= 1 \times x \times x \times x \ldots (p+q)$ multiplications

$= x^{p+q}$.

Corollary.

$x^p \times x^q \times x^r = x^{p+q+r}$ and so on for any number of factors.

177. Division Law.

If one power of a quantity be divided by a power of the same quantity, the quotient is a power whose index is obtained by subtracting the index of the divisor from the index of the dividend.

We have to show that $x^p \div x^q = x^{p-q}$.

$$x^p \div x^q = \frac{x^p}{x^q}$$

$$= \frac{1 \times x \times x \times x \times x \ldots p \text{ multiplications}}{1 \times x \times x \times x \times x \ldots q \text{ multiplications}}$$

$$= \frac{x \times x \times x \times x \ldots q \text{ factors} \times x \times x \times x \times x \ldots (p-q) \text{ factors}}{x \times x \times x \times x \ldots q \text{ factors}}$$

$$= 1 \times x \times x \times x \times x \ldots (p-q) \text{ factors}$$

$$= x^{p-q}.$$

178. Involution Law.

If a power of a quantity is itself raised to a power, the result is a power whose index is the product of the two original indices.

We have to show that $(x^p)^q = x^{pq}$.

$$(x^p)^q = 1 \times x^p \times x^p \times x^p \ldots q \text{ multiplications}$$

$$= x^{p+p+p \ldots q \text{ terms}} \quad \text{by § 176}$$

$$= x^{pq}.$$

Corollary

$\{(x^p)^q\}^r = (x^{pq})^r = x^{pqr}$, and so on for any number of successive powers.

179. To show that $(xy)^p = x^p y^p$.

By definition $(xy)^p = 1 \times xy \times xy \times xy \times xy \times \ldots p$ multiplications

$$= 1 \times x \times x \times x \times x \ldots p \text{ multiplications} \times 1 \times y \times y \times y \times y \ldots q \text{ multiplications}$$

$$= x^p y^q.$$

180. Evolution Law.

If any root is taken of any power of a quantity, the result is a power whose index is obtained by dividing the index of the original power by the index of the required root.

We have to show that $\sqrt[q]{x^p} = x^{\frac{p}{q}}$,

$$(x^{\frac{p}{q}})^q = x^{\frac{p}{q} \times q}, \text{ by § 178}$$
$$= x^p.$$

Therefore $x^{\frac{p}{q}} = \sqrt[q]{x^p}$, by definition of root, § 175,

i.e. $\sqrt[q]{x^p} = x^{\frac{p}{q}}$.

181. To show that $\sqrt[q]{x^p} = \{\sqrt[q]{x}\}^p$.

Let $\{\sqrt[q]{x}\}^p = y$.

Then $y^q = \{\sqrt[q]{x}\}^{pq} = [\{\sqrt[q]{x}\}^q]^p = x^p$;

$\therefore \sqrt[q]{x^p} = \sqrt[q]{y^q} = y = (\sqrt[q]{x})^p$.

182. The student must notice carefully what has been assumed in the preceding proofs. In all of them it is assumed that p and q are positive integers, for otherwise the definitions of § 175 have no meaning.

In § 177 it is assumed that p is greater than q.

In § 180 it is assumed that p is a multiple of q, for otherwise $x^{\frac{p}{q}}$ would have no meaning.

Let us now see what results would be obtained if we neglect these restrictions and apply the laws in all cases.

Examples XXI a.

Use the Multiplication, Division, Involution, and Evolution laws for powers to work out the following examples, regardless of all restrictions.

1. $x^3 \times x^5$.
2. $2x^3 \times 3x^5$.
3. $(2x)^3 \times (3x)^5$.
4. $x^7 \div x^2$.
5. $2x^7 \div 3x^2$.
6. $(2x)^7 \div (3x)^5$.
7. $(x^3)^5$.
8. $2(x^3)^5$.
9. $(2x^3)^5$.
10. $\sqrt[3]{x^{12}}$.
11. $8\sqrt[3]{x^{12}}$.
12. $\sqrt[3]{8x^{12}}$.
13. $x^3 \div x^5$.
14. $14x^3 \div 3x^5$.
15. $a^2b^3 \div 2ab^5$.
16. $(2a^3)^4$.
17. $(3a^2)^3$.
18. $(2ab^2)^x$.
19. $\sqrt[3]{x^5}$.
20. $\sqrt[5]{ab^4}$.
21. $\sqrt[4]{a^2b \div a^4b}$.

22. Adopting the definition of § 175, endeavour to obtain a meaning for x^0, y^{-2}.

23. What does $x^{\frac{1}{2}}$ mean according to the definition of x^n? What would $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ mean? Hence deduce the value of $25^{\frac{1}{2}}$.

24. Apply the division law to the following cases:

$$x^3 \div x^3, \quad x^7 \div x^3, \quad x^3y^4 \div x^3y^4.$$

183. It is now seen that if we use the laws of §§ 176–181 in all cases, we obtain powers with negative or fractional indices. We must, therefore, extend the meaning of x^n so as to include cases where n is not a positive integer. This may be done in several ways, all leading to the same conclusions.

184. By division law $x^3 \div x^7 = x^{-4}$.

But $$x^3 \div x^7 = \frac{x^3}{x^7} = \frac{1}{x^4};$$

$\therefore$ x^{-4} must be the same as $\frac{1}{x^4}$.

Similarly $\frac{x^q}{x^{p+q}} = x^{-p}$ by division law,

or $= \frac{1}{x^p}$ by dividing throughout by x^q;

$\therefore$ x^{-p} must mean $\frac{1}{x^p}$.

This seems a reasonable interpretation of the meaning of a negative power. For if x^p = (1 multiplied by x p times), it is to be expected that x^{-p} = (1 divided by x p times),

i.e. $= \frac{1}{x^p}$.

185. $x^p \div x^p = x^0$ by division law,

but $x^p \div x^p = 1$;

$\therefore$ $x^0 = 1$.

Or from the definition,

x^0 = 1 multiplied by x no times

$= 1$.

186. By evolution law, $\sqrt[q]{x^p} = x^{\frac{p}{q}}$.

Hence, if $\frac{p}{q}$ is a fraction, $x^{\frac{p}{q}}$ can mean nothing else than the q^{th} root of x^p, e.g. $x^{\frac{3}{2}} = \sqrt[2]{x^3}$.

It is difficult to show that this agrees with the original definition, but the following example shows how it may be done in special cases.

To find the meaning of $36^{\frac{1}{2}}$.

36 = 1 multiplied by 36
= 1 multiplied by 6 twice.

Hence, $36^{\frac{1}{2}}$ = 1 multiplied by 36 half a time
= 1 multiplied by 6 once
= 6
$= \sqrt{36}$.

Similarly, since $x = 1 \times \sqrt{x} \times \sqrt{x}$, $x^{\frac{1}{2}} = 1 \times \sqrt{x} = \sqrt{x}$.

187. We have now found reasonable meanings to attach to powers whose indices are not positive integers, but in doing so we have assumed two laws, viz. the division and evolution laws, to be true in all cases. It is desirable to assume as little as possible, and we proceed to show how these meanings may be deduced by assuming only one law.

Assume that the multiplication law is true whether the indices are fractional or integral, positive or negative, and accept the results this assumption leads to.*

188. To find a meaning for $x^{\frac{p}{q}}$ where p and q are positive integers.

By the assumption of § 187,

$$x^{\frac{p}{q}} \times x^{\frac{p}{q}} = x^{\frac{2p}{q}},$$

$$x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times x^{\frac{p}{q}} = x^{\frac{3p}{q}},$$

$$x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times x^{\frac{p}{q}} \dots q \text{ factors} = x^{\frac{p}{q}+\frac{p}{q}+\dots q \text{ terms}},$$

$$\text{i.e. } \left(x^{\frac{p}{q}}\right)^q = x^{\frac{p}{q}\times q} = x^p;$$

$$\therefore\ x^{\frac{p}{q}} = \sqrt[q]{x^p}.$$

In numerical examples the result of § 181 is useful, e.g.

$$\begin{aligned} 729^{\frac{5}{6}} &= \sqrt[6]{729^5} \\ &= (\sqrt[6]{729})^5 \\ &= 3^5 \\ &= 243. \end{aligned}$$

* Compare § 98.

To find a meaning for x^0.

By the assumption of § 187,

$$x^p \times x^0 = x^{p+0} = x^p.$$

Divide by x^p; $\quad \therefore \quad x^0 = x^p \div x^p = 1.$

To find a meaning for x^{-p} where p is any positive quantity.

By the assumption of § 187,

$$x^{-p} \times x^{+p} = x^{-p+p} = x^0 = 1.$$

Divide by x^p; $\quad \therefore \quad x^{-p} = \frac{1}{x^p}$.

189. We are now justified in giving the following definitions:

$x^{\frac{p}{q}}$ means $\sqrt[q]{x^p}$ when p and q are positive integers.

x^{-p} means $\frac{1}{x^p}$ when p is any positive quantity.

Examples XXI b.

Find the values of:

1. $8^{\frac{1}{3}}$.
2. 8^0.
3. $8^{-\frac{1}{3}}$.
4. $16^{\frac{1}{2}}$.
5. 16^0.
6. $16^{-\frac{1}{4}}$.
7. $(125)^{\frac{1}{3}}$.
8. $125^{-\frac{1}{3}}$.
9. $64^{\frac{1}{3}} - 36^{\frac{1}{2}}$.
10. $3^3 \times 9^{\frac{1}{2}}$.
11. $4^{\frac{1}{2}} \times 4^{-\frac{1}{2}}$.
12. $3^{\frac{1}{2}} \div 3^{\frac{1}{2}}$.
13. Simplify $\sqrt[3]{5} \times \sqrt{5}$ and $\sqrt[3]{\frac{1}{4}} \times \sqrt[3]{16}$.
14. Prove that $\sqrt[3]{9} \times \sqrt{8} \times \sqrt[3]{36} = 6\sqrt[3]{32}$.

Give alternative forms, not involving negative or fractional indices, for the following expressions:

15. $2x^{\frac{1}{2}}$.
16. $(2x)^{\frac{1}{2}}$.
17. $(2x)^{\frac{1}{2}} \times 2y^{\frac{1}{2}}$.
18. $(3x^{\frac{1}{2}})^3$.
19. $(3x)^{\frac{1}{2}} \times 3^{\frac{1}{2}}$.
20. $x^{-2}y^3$.
21. $9x^{-\frac{1}{2}}$.
22. $2x^{-\frac{1}{2}} \times 3x^{\frac{1}{2}}$.
23. $(3x)^{-\frac{1}{2}} \times \left(\frac{x}{3}\right)^{\frac{1}{2}}$.

24. (*a*) Find correct to two places of decimals the values of $10^{\cdot 6}$, $10^{\cdot 125}$, $10^{\cdot 0625}$. From these values find by multiplication $10^{\cdot 625}$, $10^{\cdot 5625}$, $10^{\cdot 1875}$.

(*b*) Draw the graph of $x = 10^y$ from $x = 1$ to $x = 10$. Hence find the value of y when $10^y = 8$, and what is the value of 10^y when $y = \cdot 48$? See fig. 50.

Given that $10^{\cdot 30103} = 2$ and $10^{\cdot 47712} = 3$, find x in the following cases:

25. $10^x = 20$, $10^x = \cdot 2$, $10^x = 1$.

26. $10^x = 300$, $10^x = \cdot 003$, $10^x = \frac{1}{3}$.

27. $10^x = 4$, $10^x = 5$, $10^x = 6$.

28. $10^x = \sqrt[4]{2}$, $10^x = \sqrt[4]{3}$, $10^x = 3^3 \times 2^{10}$.

Between what values must x lie in the following equations:

29. $2^x = 9$, $2^x = \cdot 9$, $2^x = 93$.

30. $10^x = 747{\cdot}2$, $10^x = \cdot 07472$, $10^x = 50$.

190. Before we can use fractional and negative powers, as defined in § 189, we ought to show that they obey the same laws as positive integral powers. We give here the proof of the involution law, and leave the proofs of the others as exercises for the student.

To prove that $(x^p)^q = x^{pq}$ for all values of p and q.

Case I when p and q are both positive integers.
This is proved in § 178.

Case II when q is a positive integer and p has any value.
$(x^p)^q = 1 \times x^p \times x^p \ldots q$ multiplications,
$= x^{pq}$ since the multiplication law is true in all cases.

Case III when q is a positive fraction and p has any value.

Let $q = \frac{h}{k}$, where h and k are positive integers.

Then $$(x^p)^q = (x^p)^{\frac{h}{k}}.$$

Now $$\{(x^p)^{\frac{h}{k}}\}^k = (x^p)^{\frac{h}{k} \times k} \quad \text{(by Case II)}$$
$$= (x^p)^h$$
$$= x^{ph}.$$

Take the k^{th} root of both sides.

Then $$(x^p)^{\frac{h}{k}} = \sqrt[k]{x^{ph}} = x^{\frac{ph}{k}} \quad (\S\ 180), \S 188.$$

i.e. $$(x^p)^q = x^{pq}.$$

Case III when q is negative and p has any value.
Let $q = -r$, where r is any positive quantity.

Then $$(x^p)^q = (x^p)^{-r} = \frac{1}{(x^p)^r} = \frac{1}{x^{pr}}$$
$$= x^{-pr}$$
$$= x^{pq}.$$

When fractional and negative powers have the meanings derived in § 188, it has now been proved that all the laws

of §§ 176–181 are true. In working examples, therefore, fractional and negative indices are dealt with exactly in the same way as positive indices. The equivalent reciprocal and root forms should generally not be used unless the question requires that the answer be given free of all indices not positive integers, and even then the index forms should be retained as long as possible.

Example I. *Simplify* $\sqrt[4]{x^2y^{-3}z^4} \times (x^{-\frac{1}{2}}y^{\frac{3}{4}}z^{-\frac{2}{3}})^3$.

$$\begin{aligned}\text{Expression} &= x^{\frac{2}{4}}y^{-\frac{3}{4}}z^{\frac{4}{4}} \times x^{-\frac{1}{2}\times 3}y^{\frac{3}{4}\times 3}z^{-\frac{2}{3}\times 3} && (\S\,180)\\ &= x^{\frac{1}{2}}y^{-\frac{3}{4}}z \times x^{-\frac{3}{2}}y^{\frac{9}{4}}z^{-\frac{6}{3}}\\ &= x^{\frac{1}{2}-\frac{3}{2}}y^{-\frac{3}{4}+\frac{9}{4}}z^{1-\frac{6}{3}} && (\S\,176)\\ &= x^{-1}y^{\frac{6}{4}}z^{-\frac{6}{3}}.\end{aligned}$$

If now the answer is required without negative and fractional indices, we have

$$\text{Expression} = \frac{\sqrt[4]{y^6}}{x\sqrt{z^3}}.$$

Example II. *Multiply* $x^{-\frac{1}{2}} - \dfrac{y^{\frac{1}{2}}}{\sqrt[3]{x}} + y^{\frac{1}{2}}$ *by* $\dfrac{1}{x^{\frac{1}{2}}} + \sqrt[3]{y}$.

[Express all roots as fractional powers.]

$$\begin{array}{l} x^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}} \\ x^{-\frac{1}{2}} + y^{\frac{1}{3}} \\ \hline x^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y^{\frac{1}{3}} \\ \qquad + x^{-\frac{1}{2}}y^{\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}} + y \\ \hline x^{-\frac{1}{2}} \qquad\qquad\qquad + y \end{array} \quad \text{Product.}$$

Examples XXI c.

Simplify the following expressions:

1. (i) $(a^{\frac{1}{2}}b^{-\frac{1}{2}})^6 \div (a^{-3}b^{\frac{1}{2}})^{-\frac{2}{3}}$; (ii) $\left(\dfrac{a^{\frac{1}{2}}}{a^{-\frac{1}{2}}}\right)$.

2. $\left(\dfrac{a^{\frac{1}{2}}}{b^{-\frac{1}{2}}}\right)^{\frac{2}{3}} \times \left(\dfrac{a^{-\frac{1}{2}}}{b^{-\frac{1}{2}}}\right)^{\frac{2}{3}} \div (ab^{-7})^{\frac{1}{12}}$.

3. $\sqrt[5]{\dfrac{a^{\frac{1}{2}} \times b^{-2} \times c^{3}}{a^{-\frac{1}{2}} \times b^{3} \times \sqrt[3]{\frac{1}{c}}}}$.

4. $\left\{\dfrac{a^{\frac{1}{2}}}{b^{-\frac{1}{2}}} \times \dfrac{a^{-1}}{b^{\frac{1}{2}}} \div \dfrac{b^{-\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}^{-24}$.

5. $\left\{(xy^2)^3 \left(\dfrac{x}{y}\right)^4\right\}^9 \div \left\{(x^3y)^4 \left(\dfrac{y}{x}\right)^5\right\}^2$.

6. $\dfrac{4^n \cdot 6^5}{2^3 \cdot 12^n \cdot 3^{-n}}$.

7. $\dfrac{5^3 \cdot 2^{\frac{1}{2}} \cdot 10^{-\frac{1}{2}}}{15^{\frac{1}{2}} \cdot 6^{-\frac{1}{2}} \cdot 4^{\frac{1}{2}}}$.

8. Multiply $a^{\frac{1}{2}} - x^{\frac{1}{2}}$ by $a^{\frac{1}{2}} + a^{\frac{1}{2}} x^{\frac{1}{2}} + x^{\frac{1}{2}}$.

9. Multiply $x^{\frac{1}{2}}y + x^2y^{\frac{1}{2}} + x^{\frac{3}{2}}y^2 + xy^{\frac{3}{2}}$ by $x^{-1}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{-1}$.

10. Divide $x^{\frac{3}{2}} + x^{\frac{3}{2}}y^{\frac{1}{2}} + y$ by $x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}}$.

11. Divide $x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{\frac{3}{2}} - 2y$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

12. If $a = b^{p+1}$ and $b = c^{p-1}$, prove that $ac = c^{p^2}$.

Express as simply as possible:

13. $\sqrt{x} \times \sqrt[3]{x}$.

14. $\sqrt{2} \times \sqrt[3]{4}$.

15. $\sqrt{9a^2bc^3}$.

16. $\sqrt[3]{8 \times 17}$.

17. $\sqrt{72}$.

18. $3 \cdot \sqrt[3]{4} + 2\sqrt[3]{108}$.

19. Simplify $\left(\dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}}\right)^{-9}$.

20. Simplify $\dfrac{2^{n+3}}{15^{-n-1}} \times \dfrac{6^{-n+2}}{5^{n+1}}$.

Surds.

191. **Roots which cannot be found exactly are called surds or irrational quantities**; thus $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[3]{x+y}$ are surds, but $\sqrt{9}$, $\sqrt[3]{8}$, $\sqrt{x^2+2xy+y^2}$ are not surds.

An n^{th} root is sometimes called a surd of the n^{th} order.

Many examples involving surds can be worked by using fractional indices, thus:

$$\sqrt[3]{3} \times \sqrt{3} = 3^{\frac{1}{3}} \times 3^{\frac{1}{2}} = 3^{\frac{5}{6}} = \sqrt[6]{243};$$

$$\begin{aligned}\sqrt[3]{3} \times \sqrt{2} &= 3^{\frac{1}{3}} \times 2^{\frac{1}{2}} \\ &= 3^{\frac{2}{6}} \times 2^{\frac{3}{6}} \\ &= (3^2 \times 2^3)^{\frac{1}{6}} \\ &= \sqrt[6]{72}.\end{aligned}$$

192. The product of the n^{th} roots of two quantities is the n^{th} root of their product.

For $\sqrt[n]{a} \times \sqrt[n]{b} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}$.

Examples. $\sqrt[3]{7 \times 11} = \sqrt[3]{7} \times \sqrt[3]{11}$, $\sqrt{7 \times 9} = \sqrt{7} \times \sqrt{9} = 3\sqrt{7}$.

The converse proposition is equally useful, viz.

The n^{th} root of the product of two quantities is equal to the product of their n^{th} roots.

Similar rules can be proved for quotients, viz. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and the converse.

Example. *Simplify as far as possible* $\sqrt{12} + \frac{1}{10}\sqrt{75} + 5\sqrt{\frac{1}{12}}$.

Ex. $= \sqrt{4 \times 3} + \frac{1}{10}\sqrt{25 \times 3} + 5\frac{\sqrt{1}}{\sqrt{4 \times 3}} = 2\sqrt{3} + \frac{5}{10}\sqrt{3} + \frac{5}{2\sqrt{3}}$

$= \frac{5}{2}\sqrt{3} + \frac{5}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5}{2}\sqrt{3} + \frac{5}{6}\sqrt{3} = \frac{10}{3}\sqrt{3}$.

193. The p^{th} root of the q^{th} root of a quantity is the pq^{th} root of the quantity.

For $\sqrt[p]{\sqrt[q]{a}} = \sqrt[p]{a^{\frac{1}{q}}} = a^{\frac{1}{pq}} = \sqrt[pq]{a}$.

Thus to find the 4^{th} root of a number, we find the square root of the square root; to find the 6^{th} root, find the cube root of the square root.

$$\sqrt[4]{49} = \sqrt{\sqrt{49}} = \sqrt{7}.$$

$$\sqrt[6]{729} = \sqrt[3]{\sqrt{729}} = \sqrt[3]{27} = 3.$$

194. To reduce a surd to an equivalent surd of another order.

An example will make the process clear.

Reduce $\sqrt[3]{4x^2y}$ *to a surd of the sixth order.*

$$\sqrt[3]{4x^2y} = 4^{\frac{1}{3}} x^{\frac{2}{3}} y^{\frac{1}{3}}$$

$$= 4^{\frac{2}{6}} x^{\frac{4}{6}} y^{\frac{2}{6}}$$

$$= \sqrt[6]{16x^4y^2}.$$

This shows that we may multiply the order of a surd by any quantity provided we multiply the index of each factor under the radical sign by that quantity.

Examples XXI d.

Simplify:

1. $\sqrt{75}-\sqrt{20}-\sqrt{48}+\sqrt{45}-\sqrt{5}$.
2. $\sqrt{147}-\sqrt{75}+\sqrt{48}$.
3. $\sqrt{50}+2\sqrt{18}-\sqrt{128}$.
4. $\sqrt[3]{24}+\sqrt[3]{768}-\sqrt[3]{96}$.
5. $\sqrt{98}-\sqrt{18}+\dfrac{28}{\sqrt{8}}$.

Find correct to three decimal places:

6. $\sqrt{2}$. 7. $\sqrt[3]{2}$. 8. $\sqrt[5]{2}$.
9. $\sqrt{3}$. 10. $\sqrt[3]{3}$. 11. $\sqrt[5]{3}$.
12. $\dfrac{\sqrt{2}}{2}-\dfrac{1}{\sqrt{2}}$. 13. $(\sqrt{3})^2-(\sqrt{2})^2$.
14. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$. 15. $(\sqrt{3}-\sqrt{2})^2$.
16. $(\sqrt{3}+\sqrt{2})^2$. 17. $(\sqrt{4}-\sqrt{3})^2$.
18. $(2\sqrt{7}-3)(2\sqrt{7}+3)$. 19. $(3\sqrt{5}-4)(3\sqrt{5}+4)$.

Simplify:

20. $(2\sqrt{x}+3\sqrt{y})(3\sqrt{x}+2\sqrt{y})$.
21. $(3\sqrt{a}-4\sqrt{b})(2\sqrt{a}+3\sqrt{b})$.
22. $(\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})$.
23. $(\sqrt{x}+\sqrt{y}+1)(\sqrt{x}+\sqrt{y}-1)$.
24. $(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})$.
25. $(\sqrt{5}+\sqrt{3}-\sqrt{2})(\sqrt{5}-\sqrt{3}+\sqrt{2})$.

What must the following quantities be multiplied by so that the products may be rational? Give the products.

26. $2\sqrt{x}-\sqrt{y}$. 27. $\sqrt{71}$. 28. $3\sqrt{5}+4\sqrt{3}$.
29. $\sqrt[3]{5}$. 30. $2\sqrt{14}-3\sqrt{7}$.

Find the square root of:

31. $a+2\sqrt{ab}+b$. 32. $4+3-4\sqrt{3}$. 33. $3+2\sqrt{2}$.
34. $8-2\sqrt{15}$. 35. $9+4\sqrt{5}$. 36. $26-2\sqrt{133}$.

37. Find the value of $3-2x+x^2$,

when (i) $x=\sqrt{3}$; (ii) $x=\sqrt{3}-\sqrt{2}$,

correct to two decimal places.

38. Find the value, correct to two decimal places, of

$$2+5x-x^3+3x^3-x^4,$$

when (i) $x=2\sqrt{3}$; (ii) $x=\sqrt{3}-1$.

39. Express

$$(\sqrt{8}+\sqrt{2})(\sqrt{8}-\sqrt{2}) \text{ and } (\sqrt{8}+\sqrt{2})^2+(\sqrt{8}-\sqrt{2})^2$$

in their simplest forms.

195. Rationalising. In numerical work, the denominator of a fraction should always be rationalized before the approximate values of the surds are substituted.

Example I. *Given that* $\sqrt{3}=1{\cdot}732$, $\sqrt{2}=1{\cdot}414$, *find, correct to three significant figures, the value of* $\dfrac{3}{3\sqrt{2}-2\sqrt{3}}$.

$$\frac{3}{3\sqrt{2}-2\sqrt{3}}=\frac{3}{3\sqrt{2}-2\sqrt{3}}\times\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$$

$$=\frac{3(3\sqrt{2}+2\sqrt{3})}{(3\sqrt{2})^2-(2\sqrt{3})^2}$$

$$=\frac{3(3\sqrt{2}+2\sqrt{3})}{18-12}$$

$$=\frac{4{\cdot}242+3{\cdot}464}{2}$$

$$=\frac{7{\cdot}706}{2}$$

$$=3{\cdot}85 \text{ correct to 3 significant figures.}$$

The student should work this example by substituting at once for $\sqrt{2}$ and $\sqrt{3}$, and convince himself that the rationalizing method is simpler. A more important reason for rationalizing is this, that the divisor is accurate so that we know exactly how many figures are correct in the answer, whereas in the other method the divisor is approximate.

Example II. *Rationalise the denominator of*

$$\frac{\sqrt{2}}{\sqrt{5}+\sqrt{3}-\sqrt{2}}.$$

$$\text{Fraction} = \frac{\sqrt{2}}{\sqrt{5}+\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{3}+\sqrt{2}}{\sqrt{5}-(\sqrt{3}-\sqrt{2})}$$

$$= \frac{\sqrt{2}(\sqrt{5}-\sqrt{3}+\sqrt{2})}{5-(5-2\sqrt{6})}$$

$$= \frac{\sqrt{2}(\sqrt{5}-\sqrt{3}+\sqrt{2})}{2\sqrt{6}}$$

$$= \frac{\sqrt{5}-\sqrt{3}+\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{15}+\sqrt{6}-3}{6}.$$

196. Quadratic Surds.

Prop. I. If $\sqrt{a}$ and $\sqrt{b}$ are both surds, and x, a, b are all rational, then $\sqrt{a}$ cannot equal $x+\sqrt{b}$.

Suppose $\sqrt{a} = x+\sqrt{b}$.

Square $a = x^2+2x\sqrt{b}+b$,

whence $\sqrt{b} = \frac{a-x^2-b}{2x}$,

or a surd equals a rational quantity, which is impossible;

$$\therefore \sqrt{a} \neq x+\sqrt{b}.$$

Prop. II. If

$$x+\sqrt{a} = y+\sqrt{b}, \text{ then } x = y \text{ and } a = b,$$

where $\sqrt{a}$ and $\sqrt{b}$ are surds and x, y, a, b are all rational.

$$x+\sqrt{a} = y+\sqrt{b}.$$

Transpose $\sqrt{a} = (y-x)+\sqrt{b}$

which is impossible by Prop. I unless $y-x = 0$, in which case $\sqrt{a} = \sqrt{b}$, i.e. $a = b$.

It follows that, if $x+\sqrt{a} = y+\sqrt{\ }$ then $x-\sqrt{a} = y-\sqrt{b}$.

Prop. III. If

$\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}$,

where $\sqrt{b}$ is a surd and a, b, x, y are rational.

$$\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y}.$$

Square $\quad a+\sqrt{b} = x+y+2\sqrt{xy}$;

$$\therefore\ a = x+y$$

$$\sqrt{b} = 2\sqrt{xy}.$$

Subtract $\quad a-\sqrt{b} = x+y-2\sqrt{xy} = (\sqrt{x}-\sqrt{y})^2$.

Take square root $\quad \sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}$.

Example I. *Find the square root of* $5-2\sqrt{6}$.

$$(\sqrt{x}-\sqrt{y})^2 = (x+y)-2\sqrt{xy}.$$

We see, therefore, that we want two numbers whose sum is 5 and product 6. These are 2 and 3.

Hence $\sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}$.

Example II. *Find the square root of* $7-4\sqrt{3}$.

$7-4\sqrt{3} = 7-2\sqrt{4\times 3}$ since $4\sqrt{3} = 2\sqrt{4}\times\sqrt{3}$.

$\therefore\ \sqrt{7-4\sqrt{3}} = \sqrt{4}-\sqrt{3} = 2-\sqrt{3}$.

Example III. *Find the square root of* $36+\sqrt{1292}$.

Here the numbers are rather large for mental work, so the following method is advisable.

Let $\quad \sqrt{36+\sqrt{1292}} = \sqrt{x}+\sqrt{y}$. (i)

Then $\quad \sqrt{36-\sqrt{1292}} = \sqrt{x}-\sqrt{y}$. Prop. II. § 196

Multiply $\quad \sqrt{36^2-1292} = x-y$, 216

i. e. $\quad x-y = 2$. 1296 (ii)

Again, squaring equation i, and equating rational parts

$$x+y = 36. \qquad \text{(iii)}$$

From ii and iii $\quad x = 19,\ y = 17$;

$$\therefore\ \sqrt{36+\sqrt{1292}} = \sqrt{19}+\sqrt{17}.$$

197. *Equations involving surds.*

The method of solving is clearly shown by the following examples:

Example I. *Solve* $\sqrt{2x+2}-\sqrt{6x+7}+3=0.$

Transpose so as to obtain a surd by itself on one side.

$$\sqrt{2x+2}+3=\sqrt{6x+7}.$$

Square $2x+2+9+6\sqrt{2x+2}=6x+7.$

Again transpose $6\sqrt{2x+2}=4x-4.$

Divide by 2 and square $18x+18=4x^2-8x+4.$ (A)

Transpose $4x^2-26x-14=0.$

Divide by 2 $2x^2-13x-7=0.$

Factorize $(2x+1)(x-7)=0.$

$$\therefore\ x=7 \text{ or } -\tfrac{1}{2}.$$

Verify. When $x=7$, L. H. S. $=\sqrt{16}-\sqrt{49}+3=4-7+3=0.$

When $x=-\frac{1}{2}$, L. H. S. $=\sqrt{1}-\sqrt{4}+3=1-2+3\neq 0.$

It is seen that $x=-\frac{1}{2}$ does not satisfy if we take the positive root in all cases.

Example II. *Solve* $\sqrt{2x+2}+\sqrt{6x+7}-3=0.$

Transpose as in Ex. I $\sqrt{2x+2}-3=-\sqrt{6x+7}.$

Square $2x+2+9-6\sqrt{2x+2}=6x+7.$

Transpose and divide by 2 $-3\sqrt{2x+2}=2x-2.$

Square $18x+18=4x^2-8x+4.$ (A)

This is the same as equation A in previous example, hence this equation has the same solution, viz. $x=7$ or $-\frac{1}{2}$.

Verify. When $x=7$, L. H. S. $=\sqrt{16}+\sqrt{49}-3=4+7-3\neq 0.$

When $x=-\frac{1}{2}$, L. H. S. $=\sqrt{1}+\sqrt{4}-3=1+2-3=0.$

And in this case $x=-\frac{1}{2}$ satisfies the equation, but $x=7$ does not, if we take the positive root in all cases.

These two examples show that, whenever both sides of an equation are squared in the process of solution, roots may be introduced which do not satisfy the original equation. In all such cases we must substitute in the original equation all the values of x obtained, and so determine which are roots of that equation.

Examples XXI e.

Rationalize the denominators of the following fractions:

1. $\frac{2}{\sqrt{7}}, \frac{3}{\sqrt{5}}, \frac{5}{\sqrt{3}}$.
2. $\frac{\sqrt{2}}{4-\sqrt{8}}$.
3. $\frac{2-\sqrt{3}}{2+\sqrt{3}}$.
4. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.
5. $\frac{2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$.
6. $\frac{3x}{\sqrt{a+x}+\sqrt{a-x}}$.

Given that $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{5} = 2{\cdot}236$, $\sqrt{7} = 2{\cdot}646$, find the values of the following expressions correct to 3 significant figures.

7. $\frac{2}{\sqrt{7}-\sqrt{5}}$.
8. $\frac{2\sqrt{7}+3\sqrt{5}}{3\sqrt{5}-2\sqrt{7}}$.
9. $\frac{2\sqrt{2}-\sqrt{3}}{\sqrt{2}+2\sqrt{3}}$.
10. $\frac{1}{\sqrt{7}+2\sqrt{5}} - \frac{1}{\sqrt{7}-2\sqrt{5}}$.
11. $\frac{3}{3\sqrt{3}-5} + \frac{5}{\sqrt{7}+\sqrt{5}}$.
12. $\frac{2\sqrt{5}-\sqrt{7}}{2\sqrt{5}+\sqrt{7}} + \frac{2\sqrt{5}+\sqrt{7}}{2\sqrt{5}-\sqrt{7}}$.
13. $\frac{1}{\sqrt{7}+\sqrt{5}+\sqrt{2}}$.
14. $\frac{\sqrt{5}}{\sqrt{7}-\sqrt{3}+\sqrt{2}}$.

Find the square roots of the following expressions:

15. $8-2\sqrt{15}$.
16. $16+6\sqrt{7}$.
17. $26+\sqrt{660}$.
18. $30-12\sqrt{6}$.
19. $73-12\sqrt{35}$.
20. $151-20\sqrt{57}$.
21. $38-12\sqrt{10}$.
22. $2\frac{1}{4}-\sqrt{5}$.
23. $2a+2\sqrt{a^2-x^2}$.

Find correct to three decimal places:

24. $\sqrt{11+6\sqrt{2}}$.
25. $\sqrt[4]{56-24\sqrt{5}}$.
26. $\sqrt[4]{17+12\sqrt{2}}$.
27. **Prove that** $\sqrt{13+2\sqrt{18}} = \sqrt{\frac{13+3\sqrt{18}}{2}} + \sqrt{\frac{13-3\sqrt{13}}{2}}$.
28. **Simplify** $\sqrt{1+\sqrt{21+12\sqrt{3}}}$.

Solve the equations:

29. $\sqrt{x-9}+\sqrt{x-16} = 7$.
30. $\sqrt{5x}-\sqrt{3(2-x)} = 1$.
31. $\sqrt{x^2-1}+\frac{1}{\sqrt{x^2-1}} = 1\frac{1}{3}$.
32. $\sqrt{x-3}+\sqrt{2x+1} = 2\sqrt{x}$.
33. $\sqrt{x+1}+\sqrt{9x+1} = 5$.
34. $\sqrt{2x+37}-2\sqrt{2x+1} = 2\sqrt{10x+14}$.

Examples XXI.

1. Divide $3x^{-\frac{1}{2}}+4x^{-\frac{1}{4}}-13x^{-\frac{1}{4}}+6$ by $x^{-\frac{1}{2}}+2x^{-\frac{1}{4}}-3$.

2. Find the square root of $4x+5+x^{-1}-12x^{\frac{1}{2}}+6x^{-\frac{1}{2}}$.

3. Find the square root of $4x^{\frac{1}{2}}-12x^{\frac{1}{3}}+13-6x^{-\frac{1}{3}}+x^{-\frac{2}{3}}$.

4. If $a^b=b^{-a}$, prove that $\left(\frac{b}{a}\right)^{\frac{b}{a}}=b^{\frac{b}{a}+1}$.

5. Simplify $\sqrt{\left\{\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}\right\}}$.

6. Find the square root of $(11\sqrt{7}+13\sqrt{5})(\sqrt{7}-\sqrt{5})$.

7. Calculate the value of $x+\frac{1}{x}$ when $x=\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

8. Multiply $x^{\frac{3}{2}}+y^{\frac{3}{2}}-x^{\frac{1}{2}}y^{\frac{1}{2}}$ by $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}$.

9. Solve
$$2x+3y=5\sqrt{3}+\sqrt{2}.$$
$$(\sqrt{3}+\sqrt{2})x+(\sqrt{3}-\sqrt{2})x=2.$$

10. Give rules for finding the product of two surds. Find the product of $\sqrt{18}$ and $\sqrt[3]{12}$.

11. Find the value of $\frac{\sqrt{245}+\sqrt{75}}{\sqrt{5}-\sqrt{3}}+\frac{\sqrt{245}-\sqrt{75}}{\sqrt{5}+\sqrt{3}}$.

12. Find the sum of $\frac{2+\sqrt{3}}{4+\sqrt{3}}$ and $\frac{1}{5+2\sqrt{3}}$.

13. Calculate, correct to four decimal places, $\frac{3\sqrt{2}+\sqrt{6}}{3+\sqrt{3}}$.

14. Show that $x^m\div x^n=x^{m-n}$ for all values of m and n.

Simplify $\left(\frac{x^4}{y^3}\right)^{-\frac{1}{2}}\times\left(\frac{x^{-\frac{1}{2}}}{x^{-\frac{1}{2}}}\right)^4\div(x^{-1}y^2z^{-3})^2$.

15. Multiply x^a+y^b by $x^{\frac{1}{a}}+y^{\frac{1}{b}}$.

16. Simplify $\sqrt[3]{a^{-\frac{1}{2}}b^{-1}}\times a^{\frac{5}{6}}b^{\frac{1}{2}}\times\sqrt{a^{-1}b^{\frac{3}{2}}}$.

17. State the fundamental laws of indices, and deduce interpretations of $x^{\frac{1}{2}}$ and $x^{-\frac{1}{2}}$.

18. Find the square root of $62+20\sqrt{6}$.

19. Find correct to 3 places of decimals the value of $\frac{4-\sqrt{3}}{4+\sqrt{3}}$.

CHAPTER XXII

LOGARITHMS

198. **Ordinary** arithmetical numbers are the sums of multiples of powers of ten thus:

$$3746 = 3\times10^3+7\times10^2+4\times10^1+6\times10^0,$$

$$37{\cdot}46 = 3\times10^1+7\times10^0+\frac{4}{10}+\frac{6}{10^2}$$

$$= 3\times10^1+7\times10^0+4\times10^{-1}+6\times10^{-2},$$

$$\cdot03746 = 3\times10^{-2}+7\times10^{-3}+4\times10^{-4}+6\times10^{-5}.$$

The connexion between ordinary numbers and powers of ten is shown very clearly in the following method of expressing numbers, which is often used in scientific work:

$$3746 = 1000\times3{\cdot}746 = 10^3\times3{\cdot}746,$$

$$37{\cdot}46 = 10\times3{\cdot}746 = 10^1\times3{\cdot}746,$$

$$\cdot03746 = \tfrac{1}{100}\times3{\cdot}746 = 10^{-2}\times3{\cdot}746.$$

Examples XXII a.

What is the highest power of ten contained in the following numbers respectively? Express each of them in a form similar to $10^3\times3{\cdot}746$.

1. 1157·3.	2. 243·97.	3. ·0893.
4. ·0008906.	5. 31570·5.	6. 7000·005.

Given that $10^{\cdot9645} = 9{\cdot}651$ and $10^{\cdot5130} = 3{\cdot}258$, express each of the following numbers as a power of 10:

7. 96·51.	8. ·03258.	9. 965·1.
10. ·0009651.	11. 32580.	12. ·9651.
13. $9{\cdot}651\times3{\cdot}258$.	14. $32{\cdot}58\div9{\cdot}651$.	15. $(96{\cdot}51)^{10}$.
16. $\sqrt[5]{96{\cdot}51}$.	17. $\sqrt{\cdot03258}$.	18. $3{\cdot}258+9{\cdot}651$.

199. Since $10^x\times10^y = 10^{x+y}$, it is seen that if numbers can be expressed as powers of 10, they can be multiplied by adding the indices of those powers. The other index laws can be used in a similar way. Also the previous section shows that, if the numbers between 1 and 10 can be put in the form 10^y, all other

numbers can be put in the same form by increasing or decreasing the index by a whole number.

An approximate method of finding y so that 10^y shall equal any number x between 1 and 10 is to use the graph of $x = 10^y$. If the student has not already drawn this graph (see Examples XXI b, 24), he should do so now, using as large a scale as possible. This graph is shown on a small scale in Fig. 50.

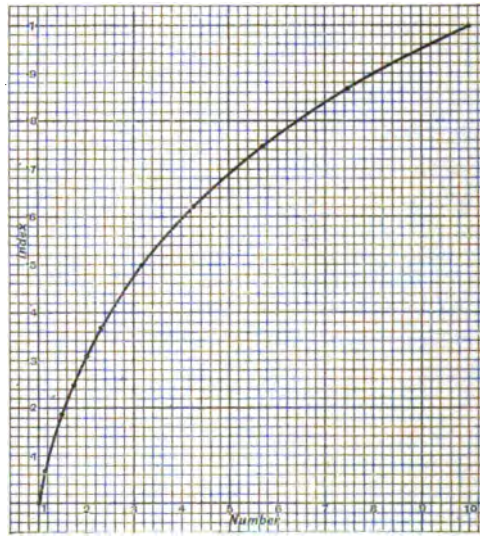

FIG. 50. Graph of $x = 10^y$.

Example. *Find the value of* $\sqrt[3]{\cdot 37} \times 9{\cdot}6^5 \div \sqrt[5]{520}$.

From the graph $3{\cdot}7 = 10^{\cdot 568}$, $9{\cdot}6 = 10^{\cdot 982}$, $5{\cdot}2 = 10^{\cdot 716}$; *

$\therefore \quad \cdot 37 = 10^{-1} \times 3{\cdot}7 = 10^{-1} \times 10^{\cdot 568} = 10^{-1+\cdot 568}$,

$520 = 10^2 \times 5{\cdot}2 = 10^2 \times 10^{\cdot 716} = 10^{2{\cdot}716}$.

Hence expression $= \sqrt[3]{10^{-\cdot 432}} \times (10^{\cdot 982})^5 \div \sqrt[5]{10^{2{\cdot}716}}$

$= 10^{-\cdot 144} \times 10^{4{\cdot}910} \div 10^{\cdot 543}$

$= 10^{-\cdot 144+4{\cdot}910-\cdot 543}$

$= 10^{4{\cdot}910-\cdot 687}$

$= 10^{4{\cdot}223}$

$= 10^4 \times 10^{\cdot 223}$.

* The third place in the index is estimated by eye and is very doubtful.

From the graph we find that when $y = \cdot 223$, $x = 1 \cdot 7$;

$\therefore$ expression $= 10^4 \times 1 \cdot 7$

$= 17000$ correct to two significant figures.

Examples XXII b.

By means of the graph of $x = 10^y$, find correct to two significant figures the values of:

1. $\sqrt[3]{4 \cdot 7}$.
2. $2 \cdot 7 \times 3 \cdot 9 \times 4 \cdot 6$.
3. $290 \times \sqrt{37}$.
4. $\dfrac{\sqrt{2} \times \sqrt[3]{3}}{\sqrt[4]{4}}$.
5. $(\cdot 079)^3 \div (\cdot 0098)^{\frac{1}{2}}$.
6. $\sqrt[3]{(490)^5}$.

200. The logarithm of a number to a given base is the index of the power to which the base must be raised to equal the given number; i. e. if $a^y = x$, y is the logarithm of x to base a. This is written $y = log_a x$.

The common logarithm of a number is the index of the power of ten which equals the number; thus $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, &c.

The logarithm of any number between 1 and 10 is more than 0 but less than 1, i. e. it is a pure decimal. A number greater than 10 can, as shown in § 198, be expressed as the product of a positive integral power of 10 and a number between 1 and 10; a positive number less than 1 can, similarly, be expressed as the product of a negative integral power of 10 and a number between 1 and 10. Hence in general the logarithm of a number consists of a decimal portion (called the **mantissa**) which is always positive, and a positive or negative integral portion (the **characteristic**).

It follows from the previous sections of this chapter that—

(i) The mantissa is independent of the position of the decimal point and depends only on the figures that make up the number.

(ii) The characteristic depends only on the position of the decimal point.

(iii) The characteristic is the index of the highest power of 10 that occurs in the number and may be found by

counting from the unit place to the first significant figure,* the unit being counted as nothing; the characteristic is positive if the number is greater than 1, negative if the number is less than one.†

Examples. $\log 3{\cdot}987 = 0{\cdot}5952$
$\log 398{\cdot}7 = 2{\cdot}5952$
$\log {\cdot}03987 = -2 + {\cdot}5952$

The negative sign of a characteristic is usually placed above the digit and the decimal written immediately after the characteristic, so that log ·03987 is written $\bar{2}{\cdot}5952$ and read 'bar 2 point 5952'.

201. The mantissa is found by means of tables. The tables at the end of this chapter give four decimal places, larger tables can be obtained giving seven or more places.

Example. *To find log* 482·3.

Counting from the 2 to the 4, the characteristic is found to be +2.

In the table on p. 294 look for 48 in the extreme left-hand column. In the same line with 48 and under 2 we find 6830, this is the mantissa of log 482; under the 3 in the small columns on the right hand and in a line with 48 we find 3, this must be added to obtain the mantissa of log 4823.

Hence log 482·3 = 2·6833.

202. The number corresponding to a given logarithm may be found from the table of logarithms by reversing the steps of the last section, or may be found from the table of anti-logarithms.

* The first significant figure is the first figure that is not nought.

† Instead of this single rule the following two rules are often given. If a number is greater than 1, the characteristic is positive and one less than the number of figures before the decimal point. If a number is less than 1, the characteristic is negative and one more than the number of noughts before the first significant figure.

Example. *To find the number whose logarithm is* 3·7603; *in other words, to find the anti-logarithm of* 3·7603.

Method I. Look in the table of logarithms for the mantissa nearest to 7603.

We find 7597 in a line with 57 and under 5, i.e. ·7597 is log 5·75.

But 7597 is too small by 6, in the right-hand columns we find that above 6 is the number 8; hence ·7603 is log 5·756.

But the characteristic is 3.

$\therefore$ 3·7603 = log $(10^3 \times 5{\cdot}756)$ = log 5756.

Method II. Look in the left-hand column of the table of anti-logarithms for 76, the first two figures of the mantissa, and proceed as in § 201.

antilog ·760 = 5·754, 3 is under 2. $\therefore$ add 2.

$\therefore$ antilog ·7603 = 5·756,

antilog 3·7603 = 5756.

203. Since logarithms are indices, the following formulae can be derived at once from the index laws:

$$\log (ab) = \log a + \log b.$$

Since $a = 10^{\log a}$, $b = 10^{\log b}$; $\therefore$ $ab = 10^{\log a \text{ and } \log b}$.

$$\log (a \div b) \text{ or } \log \frac{a}{b} = \log a - \log b,$$

$$\log (a^n) = n \log a,$$

$$\log \sqrt[n]{a} = \frac{1}{n} \log a.$$

The proofs of these are left as exercises for the student.

204. A little care is necessary when working with negative characteristics, especially when subtracting or dividing.

Example. *Find the value of* $(\cdot 03752)^{\frac{1}{3}}$.

$$\log (\cdot 03752)^{\frac{1}{3}} = \tfrac{1}{3} \log \cdot 03752$$
$$= \tfrac{1}{3} \times \bar{2}{\cdot}5742$$
$$= \tfrac{1}{3} \times \bar{8}{\cdot}8710 \qquad (-10+2=\bar{8})$$
$$= \bar{3}{\cdot}6237.$$

$\therefore$ $(\cdot 03752)^{\frac{1}{3}} = \cdot 004205.$

[The mental work for the division is as follows: The next number above 8 that is divisible by 3 is 9; also $\bar{8}{\cdot}8 = -9 + 1{\cdot}8$, therefore dividing by 3 we obtain $-3 + \cdot 6$.]

205. In working examples, the student should take care that his equalities are true throughout the work; both sides must be numbers or both sides logarithms.

Example I. *Find the value of* $\dfrac{516{\cdot}5 \times 852}{36500}$.

$$\text{Fraction} = \frac{516{\cdot}5 \times 852}{36500},$$

$$\begin{aligned}\text{log of fraction} &= \log 516'5 + \log 852 - \log 36500\\ &= 2{\cdot}7130 - 4{\cdot}5623\\ &\quad + 2{\cdot}9304\\ &= 5{\cdot}6434\\ &\quad - 4{\cdot}5623\\ &= 1{\cdot}0811.\end{aligned}$$

$$\therefore \text{ fraction} = 12{\cdot}05.$$

Example II. *Find the area of a triangle whose sides, measured in feet, are* 976·5, 394·2, 708·6.

Use the formula $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$ § 106

$$\log \text{area} = \tfrac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log s - c)\}$$

976·5	$s = 1037{\cdot}2$	$\log s = 3{\cdot}0159$
394·2	$s-a = 60{\cdot}7$	$\log s-a = 1{\cdot}7832$
708·6	$s-b = 643{\cdot}0$	$\log s-b = 2{\cdot}8082$
2|2074·3	$s-c = 333{\cdot}6$	$\log(s-c) = 2{\cdot}5232$
1037·2		sum $= 10{\cdot}1305.$

$$\begin{aligned}\therefore \log \text{area} &= \tfrac{1}{2} \times 10{\cdot}1305\\ &= 5{\cdot}0653\\ &= \log 116100.\end{aligned}$$

$$\therefore \text{ area} = 116100 \text{ sq. ft.}$$

To find log 1037·2. From tables log 1·037 is ·0158, and log 1·039 is ·0162. ∴ log 1·0372 is greater than ·0158 but less than ·0162. For small differences we use the *principle of proportional parts*, viz.

If there are 3 numbers a, b, c whose differences are small, then

$$\frac{\log b - \log a}{\log c - \log a} = \frac{b-a}{c-a} \text{ is approximately true}$$

hence

$$\frac{\log 1{\cdot}0372 - \log 1{\cdot}037}{\log 1{\cdot}038 - \log 1{\cdot}037} = \frac{{\cdot}0002}{{\cdot}001}$$

$$\therefore \log 1{\cdot}0372 = \log 1{\cdot}037 + \tfrac{2}{10} \times {\cdot}0004 = {\cdot}0159.$$

Example III. *Find the compound interest on* £2135 *for* 25 *years at* $2\frac{3}{4}$ *per cent.*

[We will first prove the compound interest formula.

£P in 1 year at r % amounts to $£P + P\frac{r}{100} = £P\left(1+\frac{r}{100}\right)$, i.e. the principal at the beginning of the year is multiplied by $1+\frac{r}{100}$ in the course of the year.

But $£P\left(1+\frac{r}{100}\right)$ is principal at the beginning of the 2nd year

$\therefore$ $£P\left(1+\frac{r}{100}\right)^2$ is amount at the end of the 2nd year, and similarly $P\left(1+\frac{r}{100}\right)^n$ is amount at the end of n years.

Using A to denote amount and R for $1+\frac{r}{100}$, we have $A = PR^n$.]

Formula is $A = PR^n$;

$\therefore$ $\log A = \log P + n \log R$

$= \log 2135 + 25 \log 1{\cdot}0275$

$= 3{\cdot}3294 + 25 \times 0{\cdot}0118$

$= 3{\cdot}3294$

$+ \cdot 295$ (The 5 is incorrect and we know nothing about the missing fourth figure.)

$= 3{\cdot}6244$ (The 44 is certainly incorrect.)

$\therefore$ $A = £4211$

and compound interest = £2076.

This answer is correct to two places, to obtain a more correct answer seven-figure logarithms should be used.

Example IV. *The volume of a right circular cone is* 176·2 *c. inches, the radius of the base is* 3·2 *inches, find the height.*

Formula is $V = \frac{1}{3}\pi r^2 h$,

i.e. $h = \frac{3V}{\pi r^2}$.

$\therefore$ $\log h = \log (3V) - \log \pi - 2 \log r$

$= \log 528{\cdot}6 - \log 3{\cdot}142 - 2 \log 3{\cdot}2$

$= 2{\cdot}7231 - 0{\cdot}4972 - 2 \times 0{\cdot}5051$

$= 2{\cdot}2259$

$-1{\cdot}0102$

$= 1{\cdot}2157$;

$\therefore$ height = 16·44 inches.

Example V. *Find the value of* $2{\cdot}718^{-3{\cdot}142}$.

Expression	$= 2{\cdot}718^{-3{\cdot}142}$	3.
log expression	$= -3{\cdot}142 \times 0{\cdot}4343$	·4343
	$= -1{\cdot}3646$	1·2568
	$= \bar{2}{\cdot}6354$	943
$\therefore$ expression	$= {\cdot}04319$	126
		9
		1·3646

[We might have found $3{\cdot}142 \times 0{\cdot}4343$ by logs instead of by ordinary multiplication.

Since the tabulated mantissae are all positive we must not leave the final logarithm as $-1{\cdot}3646$, but must actually subtract from 0, thus getting a positive mantissa and negative characteristic.]

Examples XXII c.

Find the values of:

1. $\log_2 8$. 2. $\log_8 2$. 3. $\log_2 64$.
4. $\log_9 27$. 5. $\log_2 16$. 6. $\log_2 1{\cdot}6$.

Given that $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}47712$, $\log 7 = {\cdot}84510$, find

7. $\log {\cdot}2$. 8. $\log {\cdot}036$. 9. $\log 14{\cdot}7$.
10. $\log \frac{1}{3}$. 11. $\log \frac{4}{11}$. 12. $\log 12{\cdot}348$.

By using tables find the value of:

13. $30{\cdot}56 \div 4{\cdot}105$.
14. $\cdot03056 \times 0{\cdot}4105$.
15. $\sqrt[3]{37{\cdot}24}$.
16. $254{\cdot}3 \div 0{\cdot}09027$.
17. $\sqrt{84{\cdot}05 \times 0{\cdot}1357 \div (1{\cdot}163)^3}$.
18. $(1{\cdot}342 \times 0{\cdot}01731 \div 0{\cdot}0274)^{0{\cdot}81}$.
19. $\log {\cdot}0424 \div \log {\cdot}2165$.
20. $87{\cdot}35 \div (0{\cdot}07568 \times 3{\cdot}501)$.
21. $\frac{4}{3} \times 3{\cdot}142 \times (9{\cdot}67)^3$.
22. $\sqrt{(2{\cdot}63)^4 \div (5{\cdot}834)^{\frac{1}{3}}}$.

Use tables to find the value of:

23. $\sqrt{319{\cdot}2} \times 1{\cdot}756$.
24. $\dfrac{(\cdot439)^{\frac{2}{3}} \times (14{\cdot}7)^2}{\sqrt[3]{\cdot0062}}$.
25. $3{\cdot}142 \times (71{\cdot}43)^3$.
26. $\sqrt{283{\cdot}6^3 - 245{\cdot}8^3}$.
27. $\dfrac{(\cdot1136)^{\frac{3}{4}} \times \sqrt{81{\cdot}86}}{\frac{2}{7} \times \sqrt[3]{2000}}$.
28. $(218{\cdot}3)^{-2{\cdot}86}$.
29. $\dfrac{(942{\cdot}6)^3 \times \sqrt{\cdot2}}{\sqrt[5]{7{\cdot}892 \times 10^{99}}}$.
30. $(67{\cdot}43)^{1{\cdot}414} - (56{\cdot}29)^{1{\cdot}732}$.

The mensuration formulae for the following examples are given in § 106, p. 140. In all cases the answers should be given as correctly as the data allow.

31. What is the length of the diagonal of a cube whose edge is 7·9 centimetres?

32. Find the volume of a prism whose section is an equilateral triangle of side 1·7 inches and whose length is 8·9 inches.

33. What is the area of a circular racing track 7·32 ft. wide, the outer radius being 792 ft.? Find the side of a square of the same area.

34. What is the diameter of a sphere whose volume is 2000 cubic centimetres?

35. Find the area of the curved surface of a cylinder whose volume is 597 cubic feet and length 8 feet.

36. Find the circumference of the base of a cone whose height is 1·32 ft. and volume 20 c. ft.

37. Find the amount of £215. 10*s.* invested at compound interest for 11 years at $2\frac{1}{4}$ per cent.

38. A sum of £15 is invested in 1908 to accumulate at compound interest at 3 per cent. At the end of what year will it have amounted to £50?

39. What sum will amount to £516 in 16 years at $3\frac{1}{2}$ per cent. compound interest?

40. A man wishes his son to receive £1000 in 15 years' time. If the rate of interest is $3\frac{3}{4}$ per cent., what sum must he invest now?

41. The moment of inertia I of a bar, whose length is a, is given by the formula $I = 2ma^2\dfrac{T_1^2}{T_2^2 - T_1^2}$. Find I when $a = 15{\cdot}25$, $m = 907$, $T_1 = 15$, $T_2 = 23\frac{1}{12}$.

42. If $k = \dfrac{4\pi^2 Il}{T^2}$, find k when $I = 310500$, $l = 201$, $T = 15$.

43. A quantity η is given by the formula $\eta = \dfrac{8\pi Il}{r^4 T^2}$; given $I = 3{\cdot}105 \times 10^3$, $l = 201$, $r = {\cdot}0555$, $T = 15$, find η.

44. Young's modulus Y is given by the formula

$$Y = \frac{4FL^3}{bd^3l} \times 981;$$

find Y, if $F = 200$, $l = {\cdot}326$, $L = 46{\cdot}5$, $b = 1{\cdot}089$, $d = {\cdot}518$.

45. If $\gamma = \dfrac{\log p_1 - \log p_0}{\log p_1 - \log p_2}$, and if $p_0 = 582{\cdot}4$, $p_1 = 596{\cdot}7$, $p_2 = 585{\cdot}6$, find γ.

46. If $T = 2\pi\sqrt{\dfrac{K}{MH}}$ and if $T = 22$, $K = 52{\cdot}3 \times 8{\cdot}79$, $\log \dfrac{M}{H} = 1{\cdot}5739$, find H.

206. It will be shown in a later chapter that in theoretical work the base 10 is not the most convenient base; it is, therefore, necessary to calculate the logarithms to any other required base when the logarithms to base 10 are known. For this purpose the following theorem is used.

To find the logarithms to base b when the logarithms to base a are known.

Given $\log_a N$ and $\log_a b$, it is required to find $\log_b N$.
Let $\log_b N = x$, then $b^x = N$.
Take the logs of both sides of this equation to base a.

$$x \log_a b = \log_a N$$

$$\therefore \quad x = \frac{\log_a N}{\log_b N}.$$

Example. *Find* $\log_7 15{\cdot}75$.

$$\log_7 15{\cdot}75 = \frac{\log_{10} 15{\cdot}75}{\log_{10} 7} = \frac{1{\cdot}1973}{{\cdot}8451} = 1{\cdot}417.$$

$\log 1{\cdot}1973 = {\cdot}0782,$
$\log \ {\cdot}8451 = {\cdot}9270.$
Diff. $= {\cdot}1512.$

Note.—The formulae of § 203 are true for any base.

207. By means of logarithms equations containing x as an index or part of an index can sometimes be solved.

Example I. *Solve* $8 \cdot 2^{x+3} = 192 \cdot 3^{x-3}$.

Rewrite	$3 \cdot 2^{x+3} = 3 \cdot 2^6 \cdot 3^{x-3}$.
Divide by 2^6	$2^{x-3} = 3^{x-3}$.
Take logs	$(x-3)\log 2 = (x-3)\log 3$
	$\therefore\ x = 3.$

Example II. *Solve* $3^{3x} = 7^{2x-5}$.

Take logs $3x \log 3 = (2x-5) \log 7,$

i.e. $3 \times \cdot4771\, x = (2x-5) \times \cdot8451$

$1\cdot4313\, x = 1\cdot6902\, x - 4\cdot2255.$

Transpose $\cdot2589\, x = 4\cdot2255$

$$x = \frac{4\cdot2255}{\cdot2589};$$

and the solution can be finished by ordinary division or by using logarithms.

208. Example. *The following table gives corresponding values of two quantities,* x *and* y *:*

y	10·16	12·26	14·70	20·80	24·54	28·83
x	37·36	31·34	26·48	19·08	16·33	14·04

Try whether x *and* y *are connected by a law* $yx^n = c$ *; and if so determine as nearly as you can the values of* n *and* c.

If $yx^n = c,$

then $\log y + n \log x = \log c.$

Denote $\log y$ by v and $\log x$ by $u,$

then $v + nu = \log c.$

If therefore we plot a graph for u and v, i. e. for $\log x$ and $\log y$, the graph will be a straight line.

$\log x$	1·5724	1·4961	1·4221	1·2806	1·2130	1·1473
$\log y$	1·0069	1·0885	1·1673	1·3181	1·3899	1·4599

Fig. 51 shows that the graph is almost exactly a straight line. Therefore we are justified in assuming that $\log y + n \log x = \log c.$

Substitute a pair of values

$1\cdot0069 + n \times 1\cdot5724 = \log c,$

$1\cdot4599 + n \times 1\cdot1473 = \log c.$

Subtract $-\cdot4530 + n \times \cdot4251 = 0;$

$$\therefore\ n = \frac{4530}{4251} = 1\cdot06$$

$= 1\cdot1$ to 1 decimal place.

$\log c = 2\cdot7365;$

$\therefore\ c = 550$ to 2 significant figures.

Required law is $yx^{1\cdot1} = 550.$

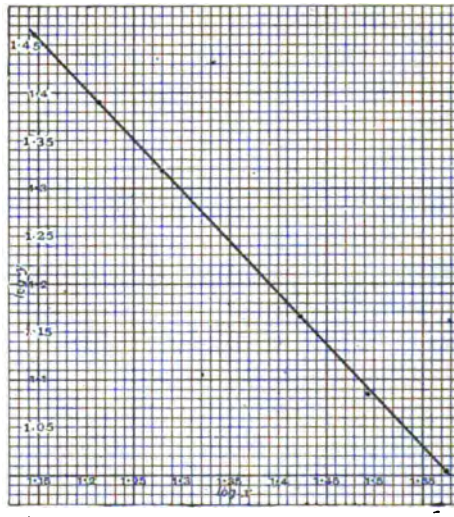

FIG. 51.

Examples XXII.

1. Prove that $\log_a b$, $\log_b a = 1$.

2. Find which of the magnitudes $\sqrt[15]{15}$ and $\sqrt[16]{16}$ is the greater.

3. Give reasons for choosing 10 as the base of logarithms. Calculate $\log_2 64$, $\log_2 640$, $\log_2 6{\cdot}4$.

4. Find (i) the number of digits and the first significant figure in 2^{10000}, (ii) the position of the first significant figure in $(\cdot 5)^{10000}$.

5. Prove that $\log_a ab = \log_a a + \log_a b$.

Given $\log 12 = x$, $\log 18 = y$, express $\log 2$ and $\log 3$ in terms of x and y.

6. Why is it that logarithmic tables give only mantissae?

Find the value of $(123{\cdot}4)^{\frac{1}{3}} \times (\cdot 007)^{\frac{1}{3}}$.

Solve the following equations:

7. $3^x = 177147$.

8. $3^{\sqrt{x}} = 243$.

9. $2^{x^2-9x-24} = 4096$.

10. $2^{x+1} + 4^x = 80$.

11. $x^2+y^2=425$, $\log x+\log y=2$.

12. $5^{3x-2y}=3125$, $11^{6x-7y}=14644$.

13. $7^{2x}-6\cdot7^x+5=0$.

14. The following quantities are thought to follow a law like $pv^n=$ constant. Try if they do and find the most probable value of n.

v	1	2	3	4	5
p	205	114	80	63	52

15. At the following draughts in sea-water a particular vessel has the following displacements:

Draught h feet	15	12	9	6·3
Displacement T tons	2098	1512	1018	586

Try to get a simple rule connecting T and h.

16. In some experiments in towing a canal boat the following observations were made; P being the pull in pounds and v the speed of the boat in miles per hour:

v	1·68	2·43	3·18	3·60	4·03
P	76	160	240	320	370

Find an approximate formula connecting P and v.

17. Prove the formula $A=P\left(1+\frac{r}{100}\right)^n$. If A is 130, when P is 100, and n is 7·5, find r.

18. If pv^k is constant, and if $p=1$ when $v=1$, find for what value of v p is 0·2. Do this for the following values of k, 0·8, 0·9, 1·0, 1·1. Tabulate your answers.

19. Calculate to 4 significant figures $2{\cdot}307^{0{\cdot}65}$ and $23{\cdot}07^{-1{\cdot}25}$.

20. $y=a+bx^n$ is the equation to a curve which passes through the three points $x=0$, $y=1{\cdot}24$; $x=2{\cdot}2$, $y=5{\cdot}07$; $x=3{\cdot}5$, $y=12{\cdot}64$; find a, b, n.

	0	1	2	3	4	5	6	7	8	9	1 2 3 4	5	6 7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 9 13 17 4 8 12 16	21 20	26 30 34 38 24 28 32 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 12 15 4 7 11 15	19 19	23 27 31 35 22 26 30 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 11 14 3 7 10 14	18 17	21 25 28 32 20 24 27 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 7 10 13 3 7 10 12	16 16	20 23 26 30 19 22 25 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9 12 3 6 9 12	15 15	18 21 24 28 17 20 23 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 9 11 3 5 8 11	14 14	17 20 23 26 16 19 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8 11 3 5 8 10	14 13	16 19 22 24 15 18 21 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 5 8 10 2 5 7 10	13 12	15 18 20 23 15 17 19 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7 9 2 5 7 9	12 11	14 16 19 21 14 16 18 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7 9 2 4 6 8	11 11	13 16 18 20 13 15 17 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6 8	11	13 15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6 8	10	12 14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6 8	10	12 14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6 7	9	11 13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5 7	9	11 12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5 7	9	10 12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5 7	8	10 11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5 6	8	9 11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5 6	8	9 11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4 6	7	9 10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4 6	7	9 10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4 6	7	8 10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4 5	7	8 9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4 5	6	8 9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4 5	6	8 9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4 5	6	7 9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4 5	6	7 8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3 5	6	7 8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3 5	6	7 8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3 4	5	7 8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3 4	5	6 8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3 4	5	6 7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3 4	5	6 7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3 4	5	6 7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3 4	5	6 7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3 4	5	6 7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3 4	5	6 7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3 4	5	5 6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3 4	4	5 6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3 4	4	5 6 7 8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3 3	4	5 6 7 8

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53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
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71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
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74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
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81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
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·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
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·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
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·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
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·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

MISCELLANEOUS EXAMPLES

1. If $x = 1$, $y = -2$, $z = 3$, find the value of

$$\tfrac{1}{2}[x - \tfrac{1}{3}\{y - \tfrac{1}{4}(z - \overline{x - y})\}].$$

2. Show that

$$[(p+q)a-(r-s)b][(p-q)a+(r+s)b]-4qrab$$
$$\equiv[(p+q)a+(r+s)b][(p-q)a-(r-s)b].$$

3. Find the factors of:

(i) $a(x^2-y^2)+b(x^2-2xy+y^2)$;

(ii) $a^2-4b^2-12b-9$;

(iii) $a^2-b^2-c^2+d^2-2bc-2ad$.

4. From two towns a miles apart, A and B started out at the same time to meet each other, A going p miles an hour and B q miles an hour. How long will it be before A and B meet? How far will they respectively have travelled?

5. Simplify:

(i) $a + \dfrac{x^2+a^2}{2x-a} + \dfrac{x^2+2ax}{a-2x}$;

(ii) $\dfrac{1}{x+2} - \dfrac{1}{x^2-4} - \dfrac{x-8}{x^2+5x+6}$.

6. Prove that $x-a$ is a factor of $f(x)$ if $f(a) = 0$.

Hence, show that $5x^4-2x^3-24x^2+8x+16$ can be divided by x^2-4.

7. Solve:

(i) $\dfrac{1}{x-1} - \dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{x+2}$;

(ii) $2x+y = z+x+3 = y+z$,
$3x+2y-4z = -6$;

(iii) $\dfrac{1}{x-1} + \dfrac{2}{x-2} = \dfrac{4}{3}$.

8. Draw the graph of (i) $y = 2x$, (ii) $x+2y = 6$.

9. Explain what is meant by the words term, variable, graph, function.

10. A and B have seventeen coins between them, which are all half-crowns or florins. A has twice as many florins as half-crowns and B has two more half-crowns than florins; also A has a shilling more than B. How much has each?

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11. If $a-b=x=3$ and $a+b+x=2$, find the value of
$$(a-b)\{x^3-2ax^2+a^2x-(a+b)b^2\}.$$

12. Divide $\frac{1}{2}a^3+\frac{3}{2}a^2x-2x^3$ by $\frac{1}{2}a+x$.

13. Simplify
$$\tfrac{1}{6}\{x(x+1)(x+2)+x(x-1)(x-2)\}+\tfrac{2}{3}(x-1)x(x+1).$$

14. Find the factors of:

(i) $ax-2a-3x+6$; (iii) $x^4+4x^3+12x-9$;

(ii) $x^2+x-600$; (iv) $a^3+3a^2b+3ab^2+b^3+c^3$.

15. Find the simplest expression that can be exactly divided by each of the following expressions:
$$a^2-ab,\ b^2+ab,\ a^3-b^3,\ a^2-b^2,\ a^3+b^3.$$

16. If $f(x)=3x^2-5x+4$, find the value of
$$f(x-2)+f(x)-f(x+2).$$

17. Solve the equations:

(i) $a(x+y)-b(x-y)=2a^2$,
$(a^2-b^2)(x-y)=4a^2b$;

(ii) $3x-[2x+4-(3x-5)]=6x+2-[3x-2-(8-2x)]$.

18. A person, having a sum of money to divide among A, B, and C, gave A one-third and £20 more, B one-third of what was left and £20 more, and C one-third of what was then left and £20 more, which exhausted the amount. How much did A, B, and C respectively receive?

19. Show by drawing graphs that the three equations
$$3x-2y=13,\ y=4x-14,\ 2x+5y+4=0$$
are satisfied by the same values of x.

20. Prove that the value of a fraction is unaltered by multiplying its numerator and denominator by the same quantity.

21. Prove that $a-(b-c)=a-b+c$, and that

$$a(x+y-z)=ax+ay-az.$$

State any restrictions to the values of the letters that you assume in your proof.

22. Multiply $\frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{3}$ by $\frac{3}{2}x^2+x+\frac{2}{3}$.

23. Find the H. C. F. and L. C. M. of

$$6y^2-y-2,\ 3y^2+5y+2,\ 3y^3-y^2+y+2.$$

24. Find the square root of $\frac{4}{9}x^2+\frac{4}{3}x-1-\frac{3}{x}+\frac{9}{4x^2}$.

25. Apply the ordinary rules for indices to prove that

$$\sqrt[y+z]{\left(\frac{\sqrt[y-z]{x^z}}{\sqrt[y+z]{x}}\right)^{y^2-1}}=x.$$

26. State and prove the rules for 'transposing'.

27. Solve:

(i) $$\frac{1-\frac{1}{x}}{1+\frac{1}{x}}+\frac{1+\frac{1}{x+2}}{1-\frac{1}{x+2}}=2;$$

(ii) $$1-a^4+\frac{a^4-x}{a^4}-\frac{a^3+x}{a^3}+\frac{a^2-x}{a^2}-\frac{a+x}{a}=0.$$

28. State and prove the formula for solving a quadratic equation.

Solve $3x^2-4x-9=0$ by two algebraical methods and one graphical method.

29. A room is such that, if it had been 5 feet longer and 3 feet narrower its area would have been unaltered, and if it had been 5 feet wider, it would have been a square. Find its length and breadth.

30. By drawing a graph find the minimum value of

$$3x^2-5x+13;$$

and verify by an algebraical method.

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31. If $a=1$, $b=-2$, $c=\frac{1}{2}$, find the value of

$$\frac{a^3}{(a-b)(c-a)}+\frac{b^3}{(b-c)(a-b)}+\frac{c^3}{(c-a)(b-c)}.$$

32. Divide

$$8x^4-18x^3y+9x^2y^2+4xy^3-3y^4 \text{ by } 2x^2-xy-y^2.$$

33. Simplify:

(i) $\dfrac{4}{x-3a}-\dfrac{x-5a}{x^2-5ax+6a^2}-\dfrac{3}{x-2a}$;

(ii) $\dfrac{x^2+a^2}{2a^2+3ax-2x^2}\div\left(\dfrac{2a+x}{2a-x}+\dfrac{a-2x}{a+2x}\right)$.

34. A room a feet long and b feet wide is carpeted in such a way as to leave a margin of c inches wide between the carpet and the walls. Find the area of the margin.

35. If ax^2+bx+c is equal to 0 when $x=1$, equal to 1 when $x=2$, and equal to 6 when $x=3$, find the values of a, b, and c.

36. Solve:

(i) $\dfrac{1}{2x-3}+\dfrac{2}{x-5}=\dfrac{12}{x+4}$;

(ii) $\dfrac{x-4}{3}-\dfrac{y-3}{4}=\dfrac{2x+5}{5}-\dfrac{1-y}{9}=2\frac{1}{3}$.

37. A and B can do a piece of work in 10 days, A and C can do it in 12 days. If the work is done by all three, and B receives £3 more than C, find how much A and B together should receive.

38. Justify the method of solving quadratics by factors.

Draw the graphs of:

(i) $(y-2x+1)(y+2x-1)=0$;

(ii) $(y-2x+1)^2+(y+2x-1)^2=0$.

39. Solve $x^2y-xy^2=6=2xy$.

40. If a man spends 22*s.* a year on tea whatever the price of tea may be, what amounts of tea will he receive when the price per lb. is 12, 16, 18, 20, 24, 28, 33, and 36 pence respectively? Give results to the nearest quarter of a lb. Draw a curve to the scale of 4 lb. to the inch and 10*d.* to the inch to show the number of pounds that he would receive at intermediate prices.

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41. Find the value of

$$(a+3b+6c)\sqrt{\tfrac{1}{8}+a^2+b^2+c^2}+\frac{4a}{c^2}\left(\frac{a-c}{b}+3abc\right)(3a+2b)^2$$

when $a=1$, $b=-2$, $c=\frac{1}{2}$.

42. Prove the laws (i) $a^m\times a^n=a^{m+n}$; (ii) $(a^m)^n=a^{mn}$.

If these laws are true for all values of m and n, what must be the meaning of $a^{\frac{1}{2}}$?

43. Find the square root of $x^4-8x^3+2x+12+\frac{1}{(x+2)^2}$.

44. (i) Find the product of

$$x^2-2x+4,\ x^2+2x+4,\ x-2,\ x+2;$$

(ii) Find the factors of

$$a^2x^2+b^2y^2+1-2ax+2by-2abxy.$$

45. Simplify:

(i) $\dfrac{1+\dfrac{x(x-y)}{1+xy}}{x-\dfrac{x-y}{1+xy}}$;

(ii) $\left(5+\dfrac{126}{x-7}-\dfrac{36}{x-2}\right)\left(5+\dfrac{36}{x+2}-\dfrac{126}{x+7}\right)$.

46. Plot the graph of $y=+\sqrt{x}$. How does it differ from the graph of $x-y^2=0$?

47. Solve:

(i) $\dfrac{x-1}{x+1}-\dfrac{x-2}{x+2}=\dfrac{x-3}{x+3}-\dfrac{x-4}{x+4}$;

(ii) $x^2+4y^2=17,\quad 3x-2y=-1.$

48. If α and β are the roots of $x^2-px+q=0$, prove that $\alpha+\beta=p$, $\alpha\beta=q$. Find the equation whose roots are α^2 and β^2.

49. Divide a straight line 6 inches long into two parts, so that three times the rectangle contained by the two parts shall equal half the square on the whole line.

50. If the value of $4ac-b^2$ is positive, prove that the value of the expression ax^2+bx+c is of the same sign for all values of x.

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51. What is meant by each of the following words:—factor, term, degree, homogeneous? Write a homogeneous expression of the third degree in x and a. Which, if any, of the following statements are obviously untrue? Give reasons.

(i) $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$;

(ii) $(a+b+c)^3=a^3+b^3+c^3+3(bc+ca+ab)+6abc$;

(iii) $(x+a)(x+b)(x+c)$
$$=x^3+x^2(a^2+b^2+c^2)+x(bc+ca+ab)+abc.$$

52. (i) Resolve x^6+27 into factors;

(ii) Resolve $36a^2+\frac{1}{4}b^2+\frac{1}{9}c^2+\frac{1}{3}bc-4ca-6ab$ into two equal factors.

53. Solve $x^2+3x=6$ correct to two decimal places and verify both solutions.

54. Solve $x+y=2$, $x^3+y^3=3\frac{1}{2}$.

55. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$, prove that $k=\frac{px-qy+rz}{ap-bq+cr}$.

56. A motor car runs from A to B by one road in $37\frac{1}{2}$ minutes. Another car goes by a different road which is 5 miles longer, and travelling 10 miles an hour quicker reaches B in 35 minutes. Find the rates and distances.

57. Draw in a single figure the graphs of

$$y=x-2, \quad y=\frac{(2x-1)(x-1)}{2x+1}.$$

58. A line 6 cm. long is divided into two parts; find, by drawing a graph, where the point of division must be in order that the product of the lengths of the two parts may be as great as possible.

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59. A glass h inches high, a inches in diameter at the top, and narrowing to a diameter of b inches at the bottom, holds $\frac{h(a^2+ab+b^2)}{132}$ pints. What will a glass hold for which $h=4$, $a=3{\cdot}7$, $b=1{\cdot}7$?

A vessel is to be made to hold a pint, to be 4·5 inches high and to have sides sloping so that the bottom diameter is 2·3 inches less than the top diameter. Find these diameters to the nearest tenth of an inch.

60. Prove that the numerator and denominator of a fraction may both be multiplied or divided by any quantity without changing the value of the fraction.

Reduce to its lowest terms $\frac{8x^3-38x^2+59x-30}{6x^3+13x^2-13x-30}$.

61. Simplify $\dfrac{\dfrac{x}{x+1}+\dfrac{x}{x-1}}{\dfrac{2}{x^2-1}}-\dfrac{4x-\dfrac{1}{x}}{2+\dfrac{1}{x}}$.

62. Solve:

(i) $x^2-(a+b)x+ab=0$;

(ii) $x^2-(a+b)x+ab=c^2$.

Justify the method of solving quadratics by factors.

63. There are two numbers each consisting of the same two digits. Their product is 3478, and one of them exceeds four times the sum of the digits by 3. What are the numbers?

64. Simplify:

(i) $\{(x^2)\sqrt[3]{(y^3)^4}\}^3 \div \sqrt[4]{(x^2)^6(y^4)}$;

(ii) $\sqrt[5]{2^8 \times (2^4)^3 \div 2^5}$.

65. Solve
$$x+2y=7,$$
$$xy-3x+2y-6=0.$$

Verify graphically.

66. Quantities x and y are known to be connected by a relation of the form $y=bx+c$. From the table given below find graphically or otherwise the values of b and c.

x	0	1	2	3	4
y	−1	2	5	8	11

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67. A square reservoir measuring a yards each way receives the drainage of b acres. It is found that a thunderstorm raised the level of water in the reservoir by c inches. What depth of water fell, if it may be supposed to have fallen uniformly over the whole area, and that all the water which fell found its way into the reservoir? Give a numerical result for the case $a=40$, $b=20$, $c=14$.

68. Simplify $\dfrac{(x+y)^5-x^5-y^5}{x^2+xy+y^2}$.

If $x^2=x+1$, show that $x^5=5x+3$.

69. State the factor theorem in words and symbols. Prove it.

The expression $x^4-8x^3+15x^2+8x-16$ has four factors, find them.

70. Show that $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

By actually solving the equation $x^2-3x+5=0$, show that the product of the roots is 5.

71. Solve
$$\left.\begin{aligned} 2x+3y &= 7 \\ 3x^2+xy+4y^2 &= 36 \end{aligned}\right\}.$$

What kind of curve does the second of these equations represent?

72. A man who walks at the rate of $4\frac{1}{2}$ miles per hour has

20 minutes start of another who walks at the rate of 5 miles per hour. When and where will he be overtaken? Solve graphically.

73. A wine merchant bought a cask of sherry for £9 and, after losing 3 gallons by leakage, sold the rest of the cask at six shillings per gallon above cost price, thereby realizing a profit of $33\frac{1}{3}$ per cent. on his whole outlay. How many gallons did the cask contain?

74. Draw the graphs of

$$y = x^3 - 3x - 4 \quad \text{and} \quad y = \frac{1}{x}.$$

Hence solve the equation $x^3 - 3x^2 - 4x = 1$.

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75. A goldsmith measures a finger, for which he is to supply a ring, by putting it into a series of graduated holes in a card. The right hole is 18 millimetres in diameter. The ring is to be circular in section, and its thickness (that is the diameter of the circular section) is to be 1·6 mm., so that the outer diameter of the ring is to be 21·2 mm. Assuming that the volume of such a ring of inner diameter p mm. and outer diameter q mm. is $\frac{\pi}{4}(p+q)(q-p)^2$, find the volume of the ring. Find also its weight and value if the metal weighs 16·4 grams per cm., and is worth 2*s.* 1*d.* per gram.

76. Simplify:

(i) $\dfrac{x^2 - 4xy - 21y^2}{x^2 - 9xy + 20y^2} \div \dfrac{x^2 - 13xy + 42y^2}{x^2 + 8xy - 40y^2}$;

(ii) $\dfrac{x+1}{(x-y)(x-z)} + \dfrac{y+1}{(y-z)(z-x)} + \dfrac{z+1}{(z-x)(z-y)}$.

77. Out of £x a man pays a bookseller's bill for £y being allowed 2*d.* in the shilling discount, and also a butcher's bill for £z being allowed 5 per cent. discount. The balance he divides equally among his m children; how much does each get?

78. Solve the equations:

(i) $(2x+3)(x-1) = 4x+6$;

(ii) $\dfrac{147}{2 + \dfrac{3}{4x}} = \dfrac{25}{12}\left(2 + \dfrac{3}{4x}\right)$;

(iii) $x + y + xy = 5, \quad xy - y = 1.$

79. Find by a graphical method the roots of the equation

$$5x^2 - 14x - 81 = 0.$$

Verify your result by calculation.

80. Define *logarithm, characteristic.* Calculate by logarithms the values of

(i) $\dfrac{293 \times 601}{359 \times 773}$; (ii) $401\sqrt{4523}$.

81. A crew which can row at the rate of 10 miles per hour on still water, finds that it takes twice as long to pull up a certain reach against stream as it does to come down the same reach: find how fast the river flows.

82. A rectangular garden is to be divided into six equal rectangular portions by three lines parallel to the sides. If two of the lines are drawn parallel to the length and the remaining one parallel to the breadth of the garden the perimeter of each portion is 224 yards; if two of the lines are drawn parallel to the breadth and the third parallel to the length the perimeter of each portion is 216 yards. Find the area of the garden.

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83. Simplify

$$(y-z)^2 + (z-x)^2 + (x-y)^2 + 2(y-z)(z-x) + 2(z-x)(x-y) + 2(x-y)(y-z).$$

Prove that the result of adding $\dfrac{x}{x+a} + \dfrac{x}{x+b} + \dfrac{x}{x+c}$ to $\dfrac{a}{x+a} + \dfrac{b}{x+b} + \dfrac{c}{x+c}$ is the same for all values of x.

84. Find values of a and b in order that

$$x^3 + (a+1)x^2 + (b-3)x - 2$$

may be divisible by the square of $x+2$.

85. Find for what values of x the expressions

(i) $\dfrac{3-5x}{8} - \dfrac{2x-1}{7} - 5x$, (ii) $\dfrac{2x+5}{3x-7}$

are always positive.

86. State what meanings are attached to $a^{-\frac{1}{2}}$ and $a^{\frac{1}{2}}$, and why.

Prove that $\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}} = \sqrt{10}$.

Solve $\sqrt{3+\sqrt{x}} + \sqrt{3-\sqrt{x}} = \sqrt{10}$.

87. Solve:

(i) $(x-1)(x-2)+(x-3)(x-4)=2(x-2)(x-3)+x$;

(ii) $\dfrac{1}{x-1}-\dfrac{1}{x+3}=\dfrac{1}{35}$;

(iii) $x^2-y^2=40, \quad x+5=6(y-1)$.

88. A person sells goods at £31. 5*s*. and gains as many pounds per cent. as the goods cost. Find the cost price.

89. The length of a rectangular field is 119 yards and its breadth is 19 yards. How much must the length be diminished, and how much must the length be increased, in order that the area of the rectangle may remain the same, while its perimeter is increased by 24 yards?

90. Graph the function $10+3{\cdot}4x-0{\cdot}6x^2$. Find its maximum value and the values of x for which it vanishes.

---

91. Prove that

$$x(a+b+c+d+\ldots)=xa+xb+xc+xd+\ldots.$$

Show that if you multiply together two numbers of m and n digits respectively, the product is a number containing at least $n+m-1$ and at most $n+m$ digits.

92. Show that

$$(a-b)^2(a^4x^6+b^4y^6)+ab(ax^2+by^2)^2\{2(a^2x^2+b^2y^2)-ab(x^2+y^2)\}=(a^2x^2+b^2y^2)^3.$$

93. State and prove the condition that x^2+ax+b may be exactly divisible by $x+c$.

Hence, or otherwise, find the condition that x^2+ax+b and x^2+px+q should have a common factor.

94. If $x^2+3x+4=y$, find what value of y will give equal roots for x. Illustrate your answer graphically.

95. Simplify the expressions:

(i) $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$;

(ii) $\dfrac{(a-b)\sqrt{x-c}}{\sqrt{(x-a)(x-b)}}+\dfrac{(b-c)\sqrt{x-a}}{\sqrt{(x-b)(x-c)}}+\dfrac{(c-a)\sqrt{x-b}}{\sqrt{(x-c)(x-a)}}$.

96. Trace the following graphs (to a scale 1 inch = 1):

(i) $y = 1{\cdot}4x - 0{\cdot}8$;

(ii) $y = 0{\cdot}25x^3$.

Hence solve the equation $x^3 = 5{\cdot}6x - 3{\cdot}2$.

97. The value of the expression $ax + bx^2$ is 3 when $x = 2$, and its value is $-\frac{1}{2}$ when $x = -\frac{1}{2}$; find the values of a and b, and draw a graph showing the values of the expression for all values of x between -5 and $+5$.

98. Calculate, by means of logarithms, the values of

(i) $\dfrac{519^2}{377 \times 803}$; (ii) $\sqrt[5]{23}$.

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99. Find, by two distinct methods, the value of

$$x^3 - 11{\cdot}8x^2 + 10{\cdot}3x - 121 \text{ when } x = 12.$$

100. In New Zealand a scheme of old-age pensions is in existence, whereby any one of more than 65 years of age is entitled to a pension of £p, provided that his income does not exceed £q. For every pound of income above £q, £1 is taken off the pension, and persons in receipt of income equal to or greater than £$(p+q)$ have no right to a pension. If n be the total number of persons of more than 65 years of age, a the percentage of those with incomes greater than £$(p+q)$ per annum, b the percentage with incomes less than £q per annum, £r the average income of the remainder, show that the annual cost of the scheme to the State is

$$\frac{n}{100}\{(100-a)p + (100-a-b)(q-r)\}.$$

101. Solve:

(i) $2(x-a)(a+b) - (x-3b)(a-b) = 8ab$;

(ii) $(5x-1)(3-x) = 8x$;

(iii) $4x^2 + y^2 = 101, \quad 2x - y = 11.$

102. Simplify:

(i) $\left(x + \dfrac{1}{y}\right)\left(y + \dfrac{1}{x}\right) - \left(x - \dfrac{1}{y}\right)\left(y - \dfrac{1}{x}\right)$;

(ii) $\left(19 - \dfrac{44}{4x-1} - \dfrac{62}{3x+4}\right)\left(23 + \dfrac{22}{3x+2} + \dfrac{124}{4x-5}\right)$;

(iii) $\sqrt{(a^{-\frac{1}{3}}b^3c^{-\frac{3}{2}}} \div \sqrt[3]{a^{\frac{1}{2}}b^4c^{-1}}$.

103. Show that if the roots of the quadratic $ax^2+bx+c=0$ are imaginary, then the expression ax^2+bx+c has for all values of x the same sign as a.

104. Draw the graph of $y=x^3$.

Employ it to show that the equation $x^3-3x+1=0$ has three real roots, and determine, roughly, their values.

105. Prove that $\log a^p = p \log a$ and $\log a - \log b = \log \frac{a}{b}$.

If $\log 2 = x$, $\log 3 = y$, find in terms of x and y the logarithms of $\frac{8}{9}$, 75, ·0015.

106. The law connecting the pressure and volume of a given mass of gas is $pv^\gamma = C$ where p is the pressure, v the volume, and γ and C are constants. If the gas is such that its pressure is doubled when its volume is compressed from 100 c.c. to 61 c.c., find the value of γ.

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*107. Un triangle a pour côtés $a=13$, $b=14$, $c=15$; calculer:

(i) la surface S;

(ii) le rayon R du cercle circonscrit donné par la formule

$$R = \frac{abc}{4S}.$$

108. De $\frac{1}{2}a^3x+\frac{1}{4}ax^3+\frac{3}{8}x^3$ retrancher $-\frac{3}{4}a^3x+\frac{1}{3}ax^3+\frac{5}{8}x^3$.

109. Déterminer m et n de telle sorte que le polynôme

$$x^4-3x^3+mx+n$$

soit divisible par x^2-2x+4.

110. Extraire la racine carrée des polynômes suivants:

(i) $a^{2m}+2a^m y^n+y^{2n}$; (ii) $1-2x+5x^2-4x^3+4x^4$.

111. Calculer les puissances et les racines suivantes:

(i) $(\sqrt[3]{\sqrt{8a^3}})^7$; (ii) $\left[\frac{c^3d}{(a+b)^{\frac{1}{2}}}\right]^{-\frac{1}{3}}$.

112. Résoudre les systèmes d'équations suivantes:

(i) $x+y=16$, $y+z=22$, $z+x=28$;

(ii) $$\frac{1}{3x-2y+1}+\frac{1}{x+2y-3}=\frac{5}{12},$$
$$\frac{1}{x+2y-3}-\frac{1}{3x-2y+1}=\frac{1}{12}.$$

* In some London University Examinations a knowledge of French and German mathematical terms is required. A few questions in these languages may, therefore, be found useful.

113. Construire la courbe représentée par l'équation

$$y = \frac{1}{1+x^3}.$$

114. Démontrer les propriétés suivantes :

(i) La caractéristique du logarithme d'un nombre plus grand que 1 contient autant d'unités moins que le nombre a de chiffres à sa partie entière ;

(ii) Le logarithme d'une racine d'un nombre égale le logarithme de ce nombre divisé par l'indice de la racine.

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115. Verwandle in ein Produkt von zwei oder mehr Faktoren :

(i) a^5+b^5; (ii) $8a^3+12a^2+6a+1$.

116. Löse die Klammern und vereinfache möglichst :

(i) $(c-b)(x+a)^2+(a-c)(x+b)^2+(b-a)(x+c)^2$;

(ii) $\left(\frac{x}{a}+\frac{a}{x}\right)\left(\frac{y}{b}+\frac{b}{y}\right)-\left(\frac{x}{a}-\frac{a}{x}\right)\left(\frac{y}{b}-\frac{b}{y}\right)$.

117. Löse die folgenden Gleichungen :

(i) $(4x-\frac{1}{2})(5x+\frac{3}{4}) = (2x+\frac{1}{2})(10x-\frac{3}{4})$;

(ii) $(x+1)^3 = x^3+1+3x(x+1)$;

(iii) $0{,}5x-y=\frac{1}{2}$; $0{,}6x+y=\frac{3}{5}$.

118. Finde eine Ungleichung von der Form $x<a$ oder $x>a$, wo a eine bekannte Zahl ist :

(i) $7x-3>4(x-1)$;

(ii) $\frac{1-4x}{12}-\frac{2+x}{9}+\frac{4-3x}{8}-\frac{5}{6}(1+x)>x$.

119. Verwandle in eine Potenz mit gebrochenem Exponenten : $\sqrt[4]{b^7}$, $\sqrt[4]{a^{4+b}}$. Vereinfache $(a^{\frac{1}{2}} \div a^{\frac{3}{5}})^{-5}$.

Berechne logarithmisch auf zwei Dezimalstellen

$$\sqrt[6]{\frac{13{,}43\times 5{,}67}{(0{,}912)^{\frac{1}{3}}}}.$$

120. Mit wieviel Ziffern wird $5^{(5^5)}$ geschrieben ?

Löse $7^{3x+3}+4^{x+3} = 7^{3x+1}+4^{x+3}$.

EXAMINATION PAPERS

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD.

Examination for Naval Cadetships. July, 1906.

1-10. Arithmetic.

11. Find the value of $2x^3-x^2-x-2$ when $x = 1.5$.

12. Resolve into factors:

$$4x^2-20x+25, \quad 2x^2-x-15, \quad (3x-2)^2-(2x-3)^2.$$

13. Simplify (i) $2(x+y)^2-(4x-3y)y-2(x^2-y^2)$;

(ii) $\dfrac{a^3-a^2}{a-a^2} \times \dfrac{a^2+a^3}{a+a^2}$.

14. Solve the equations:

(i) $\dfrac{2x-3}{5} - \dfrac{3x-2}{11} + \dfrac{1}{10} = 0$;

(ii) $3x + y = 2,$
$x + 3y = 9.$

15. A certain sum exceeds £5 by as much as £3 exceeds the half of that sum. What is the sum?

Examination for Naval Cadetships. July, 1907.

1-5. Arithmetic.

6. Multiply $1-2a+3a^2$ by $3-2a+a^2$.
Divide $6+a-16a^2$ by $3-4a$.

7. If $x = -2$, find the value of

$$\frac{1-x}{3+x} - \frac{1-x^2}{5-x^2} - \frac{2+x}{x^2}.$$

8. Resolve into simple factors:

$$6a^2+13a+6, \quad 6a^2+5a-6.$$

9. Simplify $\frac{3b-2a}{ab}+\frac{4c-2b}{bc}-\frac{3c-2a}{ca}$.

10. Solve the equation

$$\frac{2x-5}{3}+\frac{3x-7}{2}-\frac{4x-9}{4}=0.$$

11. Solve the simultaneous equations:

$$\frac{x}{2}-\frac{y}{5}=1,$$

$$\frac{8x}{3}-\frac{3y}{2}=1.$$

12. A farmer sells 6 more than one-third of his flock of sheep; then he sells 7 more than one-third of the remainder. He has still one-third of his original flock. What was the number of it?

ARMY QUALIFYING EXAMINATION.

SEPTEMBER, 1907.

[Candidates must not attempt more than SIX questions.]

1. Multiply $3\sqrt{xy^3}-xy+2\sqrt{\frac{y^5}{x}}$ by $\sqrt{x^3y}-2\sqrt{xy^3}$.

2. Divide $8a^3-b^3+c^3+6abc$ by $c+2a-b$.

3. Prove that the expressions

$$2x^3-3x^2-2x+3,\quad 4x^3-8x^2+x+3,\quad 4x^3-4x^2-5x+3$$

have a common factor.

4. Solve the equations:

$$6x+4y-2z-5=0,$$
$$3x-2y+4z+10=0,$$
$$5x-2y+6z+13=0.$$

5. Simplify the expression

$$\frac{(x+y-z)^2-w^2}{(x+y)^2-(z+w)^2}+\frac{(y+z-x)^2-w^2}{(y+z)^2-(x+w)^2},$$

and find the value of the simplified expression when

$$x=3,\quad y=-1,\quad z=4,\quad w=-6.$$

6. Solve the equation $2\sqrt{x}-\sqrt{5x+1}+\frac{1}{2}=0$.

7. Draw the graphs of the equations

$$4x+2y-17=0, \quad y=-2x^2+8x-2;$$

name the values of x and y common to both, and show clearly the directions which you take as positive along the axes of reference.

8. A man pays a bill for 5 tons of coal and 22 thousand cubic feet of gas, the whole amounting to £10. 13*s*. If the cost of coal were 8 per cent. lower per ton and that of gas $12\frac{1}{2}$ per cent. lower per thousand the bill would have been £9. 12*s*. Find the prices of coal per ton and gas per thousand.

OXFORD LOCAL EXAMINATIONS.

PRELIMINARY. 1907.

ELEMENTARY.

1. Find the quotient and remainder when

$$x^4-3x^3+x^2+4x+2 \text{ is divided by } x^2-x+1.$$

2. Find the value of

$$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-z)(y-x)}+\frac{z^2}{(z-x)(z-y)},$$

when $x=2$, $y=1$, $z=-3$.

3. Find the H. C. F. and L. C. M. of

$$x^3-x^2-x+1, \quad x^3-3x+2.$$

4. Simplify $\dfrac{2}{a-3b}-\dfrac{1}{a-2b}$, and divide the result by $a-b$.

5. Solve the equations:

(i) $(x+2)(x+3)=(x+4)(x+5)+2$;

(ii) $\dfrac{1}{x}+\dfrac{2}{x-1}=\dfrac{3}{x-2}$.

6. A man buys oranges at the rate of p a shilling and by selling them at q pence a dozen makes a profit of r per cent. Show that $25pq-36r=3600$.

7. A man has been saving annually a quarter of his income. His income is increased by £50 and now he saves a third of his income and finds that he is saving annually £25 more than before. What was his original income?

Higher.

1. Divide $a^3(b+c)+b^3(c+a)+c^3(a+b)$ by $a+b+c$.

2. Resolve into factors:

(i) $4x^2+15xy-4y^2$; (ii) $a^2-2ab+b^2-a+b$.

3. If $2b=a+c$, show that $(a-b)^2+2b^2+(b-c)^2=a^2+c^2$.

4. Solve the equation $\frac{x+2}{x+3}+\frac{x+3}{x+4}=\frac{1}{2}$.

5. (i) Divide $x^{\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}-2y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

(ii) Simplify $\frac{3+\sqrt{7}}{3-\sqrt{7}}+\frac{3-\sqrt{7}}{3+\sqrt{7}}$.

6. A man has £200 invested, partly at 4 per cent. and partly at $3\frac{1}{2}$ per cent., and his total income from these investments is £7. 12*s*. How much is invested at 4 per cent. and how much at $3\frac{1}{2}$ per cent.?

CAMBRIDGE LOCAL EXAMINATION.

Preliminary. 1907.

[Candidates can pass in Algebra by doing sufficiently well in Part I of this paper. Processes not algebraical are permitted.]

Part I.

1. Simplify $x-[y-2\{x-(y+x)+2y\}+2x]$.

2. Add the product $(a+b+1)(a-b+2)$ to the product $(a+b-1)(a-b-2)$;

and verify the result when $a=4$, $b=1$.

3. Divide $x^4+x^3-5x^2+13x-6$ by x^2+3x-2.

4. Simplify:

(i) $\dfrac{\frac{1}{a}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\left(\frac{1}{a}+\frac{1}{b}\right)}{\left(\frac{1}{a}+\frac{1}{b}\right)^2+\left(\frac{1}{a}-\frac{1}{b}\right)^2}$;

(ii) $\dfrac{x-1}{x+2}\left\{1-\dfrac{1}{x+1}-\dfrac{2}{x^2-1}\right\}$.

5. Solve the following equations, verifying your solutions:

(i) $\dfrac{x-5}{5}=\dfrac{x-4}{10}+\dfrac{x-3}{40}$;

(ii) $x+2y=3a-b, \quad 2x+y=3a+b.$

6. Divide 100 into 3 parts, such that $\frac{1}{2}$ of the first, $\frac{2}{3}$ of the second, and $\frac{3}{5}$ of the third are all equal.

Part II.

7. Resolve into factors:

(i) $a^2+ab+ac+bc$; (ii) $x^2-6x-187$;

(iii) $(x-2)^2-(x-2)$.

8. Find the H. C. F. and the L. C. M. of

$$x^3-3x-2 \quad \text{and} \quad x^3-x^2-5x-3,$$

expressing them as products of factors.

9. Solve the equations:

(i) $x^2-7x=30$;

(ii) $\dfrac{x-1}{x+1}+\dfrac{x-2}{x+2}=1$

correct to 2 decimal places.

10. A question on progressions.

11. Taking one inch as unit, draw with the same axes the graph of the lines $2x+y=2$, $2x+6=3y$, $y=1$; and find from your figure the co-ordinates of the three points at which the lines intersect.

COLLEGE OF PRECEPTORS.

THIRD CLASS. CHRISTMAS, 1907.

1. Rearrange the following expression in ascending powers of x, and show, by means of brackets, what are the coefficients of the different powers:

$$3bx^3-7ax-ax^3-6b-2bx^2+bx^3-1-7ax^2.$$

Find also the value of the expression when $a=2$, $b=0$, $x=-1$.

2. Multiply $7p^5-2p^4q-6p^3q^2$ by $3p^3-pq+q^2$.

3. Divide $x^6-2x^3y^3+y^6$ by x^2+xy+y^2.

4. Without more working than is necessary, calculate the coefficient of x^3 in the product of

$$x^4-29x^3-20x^2+5 \text{ by } 2x^3-x^2-x+1.$$

5. State which of the following expressions cannot be resolved into factors, and write down the factors of such as can be so resolved:

$$2a+2b,\quad a^2-b^2,\quad a^2+b^2,\quad a^3-b^3,\quad a^3+b^3,\quad a^4-b^4,\quad a^4+b^4.$$

6. Find the H. C. F. of $2x^3-9x+7$ and $3x^3-5x+2$;

and the L. C. M. of $(x+1)^3$, $(x-1)^3$, x^2-1.

7. Simplify the expressions:

(i) $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{a-b}{ab}$;

(ii) $\dfrac{a}{x-y}+\dfrac{b}{x+y}-\dfrac{ax-by}{x^2-y^2}$;

(iii) $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)\div\left(\dfrac{y}{x-y}-\dfrac{x}{y-x}\right)$.

8. Solve the equations:

(i) $\frac{3}{8}x-5=\frac{5}{12}x+\frac{1}{4}$;

(ii) $a^2x-c=b^2x+d$;

(iii) $\dfrac{3x-1}{7}+\dfrac{6-x}{4}-\dfrac{x-2}{6}=2-\dfrac{x+2}{28}$.

9. If two men start together in the same direction, walking b and c miles per hour respectively, c being greater than b, find the distance between them after d hours.

Also, if they start at a minutes after noon, how many minutes will the faster man be in walking x miles, and by what o'clock will he accomplish that distance?

10. Boys equal in number to one-fifth of a form are promoted out of it and twice as many promoted into it; also four new boys join the form. It is then found that the number of boys in the form has been increased by one-third. Find the numbers in the old and new forms.

Second Class or Junior. Christmas, 1907.

1. Write down the following products:

(i) $(x-2y)(2x-3y)$; (ii) $(3x+y)(3x-4y)$;

(iii) $(2x+5y)(x+y)$; (iv) $(a+b)(a-b+c)$.

2. Find the square root of

$$4x^4+8xy^3-4x^3y-15x^2y^2+16y^4.$$

3. Find the H. C. F. of

$$1-x^2+2x^3-x^4 \quad \text{and} \quad 1-3x+4x^2-3x^3+x^4.$$

4. Resolve into factors:

(i) $x^2+20x-96$; (ii) $x^2-20x+96$;

(iii) $2x^3y-8y^3$; (iv) $x^2+2xy+y^2-x-y$.

5. Solve the equations:

(i) $\frac{2}{3}(x-6)-\frac{2x-13}{5}=\frac{x}{3}-3$;

(ii) $\frac{2x-y}{7}-\frac{2y-x}{5}=0, \quad 4x-5y=-9$;

(iii) $2x+y=15, \quad 3y-z=13, \quad x+y-5z=-87$.

6. Simplify:

(i) $\frac{-x+5}{2x^2-5x+2}-\frac{x-5}{2x^2+x-1}+\frac{x-7}{x^2-2x-3}$;

(ii) $\frac{a^2-b^2-c^2+2bc}{a^2-b^2+c^2+2ac}$.

7. (i) A rectangular lawn x feet wide and y feet broad is surrounded by a path z feet wide. What is the area of the path?

(ii) Apples cost x shillings a score. How many will be obtained for a sovereign? How many more will be obtained if the price is lowered a shilling a score?

(iii) A cistern can be filled by one tap in x minutes, by another in y minutes. If both are opened, in what time will it be filled?

8. Solve the equation $\dfrac{1}{x-2} - \dfrac{1}{x-3} = \dfrac{-1}{4x-2}$.

The numerator of a fraction is greater than the denominator by 7; if the numerator and denominator are each increased by 18, the two fractions differ by 2. Find the fraction.

9. A train starts from a station X at midday and arrives at Y at 4 p.m. Another train starts from Y at 1 p.m. and arrives at X at 3.30 p.m. They meet 28 miles from X. Find the distance from X to Y.

Either,

10 A. A man can make a articles of one kind in a week. He can make b articles of another kind in a week. How many can he make in a week if he make the same number of each?

Or,

10 B. Draw a graph of $4-x-x^2$, and find from it the greatest value of the expression.

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD.

EXAMINATION FOR SCHOOL CERTIFICATES. JULY, 1906.

ALGEBRA.

[In order to pass in Elementary Mathematics, Candidates must satisfy the Examiners in Part I.]

PART I.

1. Add together $(a-b)^4$ and $4a(a+b)^3$, and subtract from your result $a^2(5a+3b)(a+b)$.

2. Find the H.C.F. of x^3+3x^2+16 and x^5-5x^2-48, and the L.C.M. of $18(x^3-6x^2+11x-6)$ and $12(x^3-7x+12)$.

3. Solve the equations:

(i) $\dfrac{1}{x-1}-\dfrac{2}{x-2}+\dfrac{1}{x-3}+\dfrac{x}{(x-1)(x-2)(x-3)}=0$;

(ii) $x^2+28x=165$.

4. Prove that, if $x+y=1$,

$$x^3(y+1)-y^3(x+1)-x+y=0.$$

5. Simplify the expressions:

(i) $$\frac{\dfrac{1-x}{x}+\dfrac{4}{1-x}}{\dfrac{1+x^2}{x^2}-\dfrac{4}{1+x^2}};$$

(ii) $$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}-\frac{a+b}{(c-a)(c-b)}.$$

6. Round the edge of a rectangular court which is half as long again as it is broad runs a path 3 feet wide. If the area of the path is 1584 square feet, find the dimensions of the court.

[The following questions are to be attempted only by Candidates who do not attempt Part II.]

A. Resolve into factors:

(i) x^4-10x^2+9;

(ii) $a^3-a^2b+ab^2-b^3$.

B. Solve the equations:

$$x+y=2a+b,$$
$$3x+2y=7a+3b.$$

C. Prove that, if a, b, c are three consecutive whole numbers,

$$a^3+c^3=2b(b^2+3).$$

D. Draw the graph of the function $3x^2-10x+1$ from $x=-1$ to $x=4$, and determine from your figure the values of x for which the value of the function is -4.

Higher Certificate. July, 1906.

[In order to pass in Elementary Mathematics, Candidates must satisfy the Examiners in Part I.]

Part I.

1. Find the value when $a=2$, $b=-1$, $c=-2$, of:

(i) $3c-2ac^2+\dfrac{4c^4}{ab}$;

(ii) $\dfrac{(b^2+1)^2-2ab+1-a^2-b^2}{a+b+b^2+1}$.

2. Multiply $a^2+9b^2+c^2+3bc+ac-3ab$ by $a+3b-c$.
Divide $3a^4-5a^3b+6a^2b^2-3ab^3+b^4$ by a^2-ab+b^2.

3. Simplify:

(i) $4(4x+1)^2-6(4x-1)(x-1)-4(x+1)(10x-1)$;

(ii) $\dfrac{bc(a+b)}{(a-b)(a-c)}+\dfrac{2abc}{(b-c)(b-a)}+\dfrac{ab(b+c)}{(c-a)(c-b)}$.

Find the H. C. F. of $20a^4-3a^3b+b^4$ and $64a^4-3ab^3+5b^4$.

4. Draw the graphs of $x=1+y$ and $2x+4y=17$ in the same diagram. Hence solve the two equations simultaneously. [Take 1 inch as your unit.]

5. Solve the equations:

(i) $(x+5)(x-3)=(x-5)(x+3)+4$;

(ii) $ax-by=2(a^2-b^2)$, $x-2y=-3b$;

(iii) $3x^2+x-2=0$.

6. A man buys a certain number of photographs for £1. Two get damaged, and by selling the remainder for 2*d.* each more than they cost he makes one shilling profit. How many did he buy?

[The following questions are to be attempted only by Candidates for Certificates who do not attempt Part II.]

A. Factorize:

(i) x^3-y^3; (ii) $ab(x^2-y^2)+xy(a^2-b^2)$.

B. Find the H.C.F. of x^3+x^2-2 and x^3+2x^2-3.

Show that $(x+y+z)^3-x^3-y^3-z^3=3(y+z)(z+x)(x+y)$.

C. Solve:

(i) $(a^2+x)(b^2+x)=(ab+x)^2$;

(ii) $a(x^2+1)=x(a^2+1)$.

D. A is 8 years older than B; 10 years ago he was twice his age. Find their present ages.

CENTRAL WELSH BOARD.

Elementary Mathematics, Junior Certificate.

July, 1906.

1. Find the value, when $a=-\frac{1}{3}$, $b=2$, $c=-\frac{1}{2}$, of

$$\frac{2a}{(2a+b)(2a+3c)}-\frac{b}{(b+2a)(b-3c)}-\frac{3c}{(3c+2a)(3c-b)}.$$

2. Simplify:

(i) $(a+b+c)(bc+ca+ab)-(b+c)(c+a)(a+b)$;

(ii) $\left(\frac{2}{x-3}-\frac{1}{x-2}\right)\left(\frac{1}{x-1}+\frac{1}{x-3}\right)$.

3. Resolve into factors the following expressions:

(i) $(2x-a)^2-(x-3a)^2$;

(ii) $x^3-x-2(x^2-1)$;

(iii) $p^2-2pq+q^2-3p+3q$.

4. Simplify $\frac{1}{x-2}-\frac{1}{x-1}-\frac{1}{(x-1)^2}$.

5. Solve the equations:

(i) $3(x-1)+5(x-2)+\frac{7x+8}{3}=0$;

(ii) $\frac{x-a}{b}+\frac{x-b}{a}=2$.

6. Solve the equations $3x-4y=1$, $2y-x=1$ both graphically and algebraically.

7. A man saves annually one-third of his income. When his income is increased by £100 a year, he only saves one-fourth, but finds that he is saving annually £12. 10*s.* more than before. What was his original income?

8. A man buys two horses for £70. By selling one of them at a profit of 10 per cent., he gains altogether £10. What did he pay for each?

Elementary Mathematics, Junior Certificate.

July, 1907.

1. Multiply together $a^3-3ab^2+2b^3$ and $a^3-3a^2b+4b^3$, and divide the product by $a^4-5a^2b^2+4b^4$.

2. Resolve into factors:

(i) $2x^3-3xy-2y^3$;

(ii) $(x-2)^3-(x-2)$;

(iii) $a(a+1)-b(b+1)$.

3. Find the H. C. F. of x^4+x^2+1 and $x^4+2x^3+x^2-1$; and find the L. C. M. of these two expressions and

$$x^5-x^4-x^3+3x^2-3x+1.$$

4. Simplify:

(i) $\dfrac{1}{a-1}-\dfrac{1}{a+1}+\dfrac{2}{a^2+1}$;

(ii) $\dfrac{b(a+b)+c(a+c)}{(a+b+c)^2-(a-b+c)(a+b-c)}$.

5. Solve the equations:

(i) $\frac{1}{2}(x-6)+\frac{2}{3}(x-2)+\frac{4}{5}(x-1)=4x-17$;

(ii) $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{2}{x}$.

In the case of the second equation, verify your solution.

6. Show, by drawing graphs, that the values of x and y which satisfy the equations $x+2y=1$, $3x-y=2$, satisfy also the equation $2x-3y=1$.

7. A man starts to bicycle from A to B at the rate of 12 miles an hour. When he has completed half the journey his bicycle breaks down and he starts to walk, doing the next 2 miles in half an hour. He is now picked up by a conveyance, which takes him the rest of the way at the rate of 8 miles an hour. If the whole journey occupies an hour and a half, what is the distance from A to B?

8. A grocer has two kinds of tea. When mixed in the proportion of 5 : 1 the mixture is worth 1*s.* 11*d.* a pound; when mixed in the proportion of 1 : 2 the mixture is worth 1*s.* 8*d.* a pound. What is the value of a pound of each kind of tea?

Practical Mathematics. July, 1906.

[1. Explain the nature and use of the slide rule.]

2. Determine the factors of:

(i) a^3-b^3; (ii) $3x^2-13x-10$;

(iii) $a^2-2ab+b^2-ac+bc$.

3. Prove that $\log ab = \log a + \log b$.

Take out of the tables the logarithms of 327 and ·00327.

Use tables to find approximately the value of $50\frac{1}{2}$, $(\frac{9}{11})^5$.

4. Draw the graphs of $2x+1$ and x^2 between the values ± 4 of x.

Use the graphs to find, correct to the first place of decimals, the roots of $x^2-2x-1=0$.

5. At noon on Monday one clock, which loses 10 minutes a day, is an hour fast, and a second clock, which gains a quarter of an hour a day, is 20 minutes slow. When will they both indicate the same time?

Draw graphs to represent the variation of each clock, and by means of the graphs verify your solution.

[There were also questions in Arithmetic and Geometry.]

Additional Mathematics. July, 1907.

1. If y is a function of x, and corresponding to the values of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 of x the observed values of y are 10, 9·9, 9·8, 9·5, 9·1, 8·7, 8, 7·1, 6, 4·4, 0 respectively, represent y graphically for values of x between 0 and 10, and find the values of y when $x =$ (i) 4·5, (ii) 8·5.

2. Find the condition that the roots of the quadratic equation $ax^2+bx+c=0$ should be real.

If α and β are the roots of the equation

$$x^2-3x+1=0,$$

find the equation whose roots are $\dfrac{\alpha}{1+\beta}$, $\dfrac{\beta}{1+\alpha}$.

3. Solve the equations:

(i) $xy=1$, $x^2-y^2=1\frac{31}{36}$;

(ii) $\sqrt{4-x}+\sqrt{6+x}=\sqrt{6(x+1)}$.

4. (i) Find the square root of $5+\sqrt{21}$.

(ii) Simplify $\dfrac{\sqrt{5}+2}{\sqrt{5}-2}\div\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.

5. Interpret, justifying your interpretation, a^0, $a^{-\frac{p}{q}}$.

Simplify $(a^{\frac{1}{2}}b^{\frac{2}{3}})^{-2}\times(b^{-\frac{2}{3}}c^{-\frac{1}{3}})^{-3}\div(a^2b^{-5}c^2)^{-\frac{1}{2}}$.

6. Prove that $\log\dfrac{a}{b}=\log a-\log b$.

If $\log_{10}2=x$, $\log_{10}3=y$, $\log_{10}7=z$, express in terms of x, y, and z the logarithms to the base 10 of 125, ·147, $\sqrt[3]{\cdot 002205}$.

[7, 8, and 9 were beyond the scope of this book.]

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SCHOOL ALGEBRA

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CHAPTER XXIII

HOMOGENEOUS EXPRESSIONS. CYCLIC ORDER AND SYMMETRY. INDETERMINATE COEFFICIENTS. HIGHEST COMMON FACTOR. SQUARE ROOT. CUBE ROOT. MISCELLANEOUS EQUATIONS

This chapter is in the nature of a revision chapter and treats of various theorems and methods of work which were regarded as too advanced in character for insertion in Part I.

209. Homogeneous Expressions. An expression in which all the terms are of the same dimensions is called a **Homogeneous expression.** Thus

$$x^2+xy+2y^2,\quad a^3-8abc+b^3+c^3,$$

are homogeneous. It may sometimes be necessary to mention what letters are regarded as giving dimensions, e.g. $x^3+ax^2y+2xy^2+3y^3$ is homogeneous in x and y, but is not homogeneous if a is regarded as a dimension.

It is clear that a homogeneous expression remains homogeneous when it is multiplied by another homogeneous expression or when its terms are rearranged. This supplies a useful test for accuracy in work.

Example. *Find the cube of* $(3a+b-c)$.

Suppose this example were worked as follows

$$\begin{aligned}(3a+b-c)^3 &= (9a^2+6ab-6ac+b^2-2bc+c^2)(3a+b-c),\\ &= 27a^3+27a^2b-27a^2c+9ab^2-18abc-6ab^2c+9ac^2\\ &\qquad +b^3-3b^2c+3bc-c^3.\end{aligned}$$

Here $a+b-c$ is homogeneous of one dimension, therefore $(a+b-c)^3$ must be homogeneous of three dimensions; but in the expanded form $-6ab^2c$ is of four dimensions, $3bc$ is of two dimensions. Therefore those two terms must be wrong.

210. Cyclic Order. So far it has always been understood that the letters are arranged in alphabetical order, the higher powers of each letter coming before the lower powers, so that, for instance, a^2b comes before ab^2 and ab^2 before abc. It is often more convenient to arrange the letters in cyclic order. We suppose the letters arranged in a circle (Fig. 52), and then in any expression the letters are arranged in the same order as in going round the circle, e. g. a before b, b before c, c before a. Thus in $(a-b)(a-c)(b-c)$ the letters are arranged in alphabetical order, if arranged in

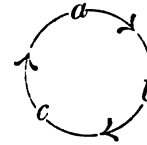

FIG. 52.

cyclic order the expression would be $-(b-c)(c-a)(a-b)$; the $-$ sign being introduced because $(a-c) = -(c-a)$. Notice that $b-c$ is placed first as being the factor that leaves out a, then $c-a$ as leaving out b, and lastly $a-b$ which leaves out c; also $c-a$ is written instead of $a-c$, because c comes before a as one goes round the circle.

Example. *Arrange* $a^2+2ab+2ac+b^2+2bc+c^2$ *in cyclic order.*
Expression $= a^2+b^2+c^2+2bc+2ca+2ab$.

211. Symmetry. If an expression is unchanged in value when two of its letters are interchanged it is said to be symmetrical with regard to those letters.

Examples. $a+b$, $2a^2-3b^2$, $a^2b+a^2c+b^2a+b^2c$, become respectively $b+a$, $2b^2-3a^2$, $b^2a+b^2c+a^2b+a^2c$, when the letters a and b are interchanged. It is seen that the first and third of these are unaltered in value although the order of the terms is changed. They are, therefore, symmetrical with regard to a and b; but $2a^2-3b^2$ is not symmetrical.

An expression may be symmetrical with regard to three or more letters, e. g. $a+b+c$, $a^3+b^3+c^3-3abc$, are both symmetrical with regard to the three letters a, b, and c.

Some expressions change sign when two letters are interchanged, but do not change sign when each letter is replaced by the letter that follows it in the circle, e. g.

$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$

becomes $b^2(c-a)+c^2(a-b)+a^2(b-c)$ when such a change is made. In such expressions there is said to be **cyclic symmetry**, although there is not **absolute symmetry**. In future we shall call an expression symmetrical if there is either absolute or cyclic symmetry.

212. The Symbols Σ and Π. Σ is the Greek capital S and is used to denote 'sum of all terms like'. It is called *sigma*.

Π is the Greek capital P and is used to denote 'product of all factors like'. It is called *pi*.

Thus, if it is known that the letters a, b, c occur symmetrically, $\Sigma(a)$ means the sum of all terms like a, i. e. $\Sigma(a) \equiv a+b+c$. Similarly $\Pi(a) \equiv abc$.

$$\Sigma(b-c) \equiv (b-c)+(c-a)+(a-b).$$

For, if we wish to obtain the terms like $b-c$, we move round the circle putting c wherever there was b, a where there was c, and b where there was a.

Examples XXIII a.

Arrange the following expressions in cyclic order:

1. $ab^2+2abc+ac^2+b^2c+bc^2+a^2b+a^2c$.
2. $x^3+3x^2y+3x^2z+3xy^2+6xyz+3xz^2+y^3+3y^2z+3yz^2+z^3$.
3. $a^2+3ab+4ac+3b^2+4bc+2c^2$.
4. Write a symmetrical homogeneous expression of the second degree in a, b, c.
5. What must be added to ab^2+2bc to make the result (i) symmetrical with regard to a and b, (ii) symmetrical with regard to a, b, and c?
6. Write down in factors the simplest symmetrical expression of the fourth degree that has $a+b-c$ as a factor.

Write in full:

7. Σa^2b, when there are (i) two letters, (ii) three letters.

8. $\Sigma x^2 - 2\Sigma(yz)$, when there are (i) three letters, (ii) four letters, x, y, z and u.

9. $\Sigma x^3 + 3\Sigma x^2y + 6\Sigma xyz$.

10. $\Pi(a+b-c)$, there being three letters.

Prove the following identities:

11. $\Sigma(y-z) \equiv 0$.

12. $\Sigma\{x(y-z)\} \equiv 0$.

13. $\Pi(b-c) \equiv -\Sigma\{a^2(b-c)\} \equiv -\Sigma\{bc(b-c)\} \equiv \Sigma\{a(b^2-c^2)\}$.

14. $\Sigma\{a^2(b^2-c^2)\} \equiv 0$.

15. $(a+b+c)^3 \equiv \Sigma a^3 + 3\Sigma(a^2b) + 6abc$.

213. A symmetrical expression can be arranged in a number of ways so that it appears to be unsymmetrical; thus

$$a+b+c \equiv (a+2b+c) - b.$$

But it will still be unaltered in value if two of the letters are interchanged, hence if an expression, symmetrical in three letters, can be put into an unsymmetrical form, two similar unsymmetrical forms can at once be written down.

Example. *Show that* $(y+z-x)$ *is a factor of*

$$(x+y+z)^3 - 4(x+y+z)(yz+zx+xy) + 8xyz,$$

and write down the other factors.

$$\begin{aligned}
\text{Expression} &\equiv \{(y+z)+x\}^3 - 4\{(y+z)+x\}\{x(y+z)+yz\} + 8xyz, \\
&\equiv (y+z)^3 + 3(y+z)^2x + 3(y+z)x^2 + x^3 + 8xyz \\
&\qquad - 4(y+z)^2x - 4(y+z)x^2 - 4yz(y+z) - 4xyz, \\
&\equiv (y+z)^2(y+z-x) - x^2(y+z-x) - 4yz(y+z-x), \\
&\equiv (y+z-x)\{(y+z)^2 - x^2 - 4yz\}.
\end{aligned}$$

[But the original expression is symmetrical, and has been put into an unsymmetrical form. Interchanging x and y, we obtain a form having $x+z-y$ as a factor; interchanging x and z, we obtain $y+x-z$ as a factor. All this is usually expressed shortly as follows.]

But the original expression is symmetrical, therefore by symmetry $z+x-y$ and $x+y-z$ are also factors. By comparing the coefficients of x^3 it is seen that the negative sign must precede the three factors.

214. The following examples show how the principle of symmetry and the use of cyclic order may be employed to shorten work.

Example I. *Find the factors of* $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

The expression is symmetrical, homogeneous, of fourth degree.

Put $b=c$, expression $=0+b^3(b-a)+b^3(a-b)=0$;

$\therefore$ $b-c$ is a factor;

$\therefore$ by symmetry $(c-a)$, $(a-b)$ are also factors.

There must be a fourth factor of the first degree since the original expression is of four dimensions; also that factor must be symmetrical.

The remaining factor is therefore $a+b+c$.

There may be a numerical factor, call it N;

then $a^3(b-c)+b^3(c-a)+c^3(a-b)\equiv N(a+b+c)(b-c)(c-a)(a-b)$.

On the right-hand side the coefficient of a^3 is $-N(b-c)$; comparing this with the left-hand side it is seen that the identity is satisfied when $N=-1$.

Or put $a=2$, $b=1$, $c=0$ then $8-2=N(3)(-2)(1)$,

$\therefore$ $N=-1$.

Hence the factors are $-(a+b+c)(b-c)(c-a)(a-b)$.

Example II. *Simplify*

$$\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}.$$

First arrange in cyclic order.

$$\text{Ex.}\quad =-\frac{1}{(a-b)(c-a)(x+a)}-\frac{1}{(b-c)(a-b)(x+b)}-\frac{1}{(c-a)(b-c)(x+c)}.$$

The expression is now symmetrical and the second fraction can be obtained from the first by **cyclic permutation**, i.e. by changing a into b, b into c, c into a, and the second fraction can similarly be changed into the third.

Reduce to common denominator.

$$\text{Ex.}\quad =-\frac{(b-c)(x+b)(x+c)+\ldots+\ldots}{(b-c)(c-a)(a-b)(x+a)(x+b)(x+c)}.$$

There is no need to fill in the second and third terms as they are obtained from the first by cyclic permutation.

Arrange the numerator according to powers of x.

Ex. $= -\dfrac{x^2\Sigma(b-c)+x\Sigma(b^2-c^2)+\Sigma\{bc(b-c)\}}{\text{Denominator}}$,

but $\Sigma(b-c)=0,\ \Sigma(b^2-c^2)=0,$

$\Sigma bc(b-c)=-(b-c)(c-a)(a-b),$

see Ex. XXIII a, Nos. 11, 13.

$\therefore$ Ex. $= \dfrac{1}{(x+a)(x+b)(x+c)}.$

Example III. *Prove the identity*

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}\equiv 1.$$

Put $x=a$, then left-hand side $=1+0+0$; $\therefore x=a$ satisfies the equation.

Similarly $x=b$, $x=c$, also satisfy, since the left-hand side is symmetrical.

Hence an equation of the second degree has more than two roots, which is impossible unless the equation is an identity (Part I, § 148).

$\therefore$ Given equation is an identity.

Examples XXIII b.

Resolve into factors:

1. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
2. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.
3. $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$.
4. $\Sigma\{x^3(y-z)\}$.
5. $(y-z)^3+(z-x)^3+(x-y)^3$.

Simplify:

6. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-z)(y-x)}+\dfrac{1}{z(z-x)(z-y)}$.
7. $\Sigma\left\{\dfrac{a^2}{(a-b)(a-c)(x+a)}\right\}$.
8. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.
9. $\Sigma\dfrac{a^2}{(a-b)(a-c)}$.
10. $\Sigma\dfrac{a^3}{(a-b)(a-c)}$.
11. $\Sigma\dfrac{a^4}{(a-b)(a-c)}$.
12. $\Sigma\dfrac{1+a+a^2}{(a-b)(a-c)}$.

Prove the identities:

13. $\dfrac{(a-x)(a-y)}{(a-b)(a-c)} + \dfrac{(b-x)(b-y)}{(b-c)(b-a)} + \dfrac{(c-x)(c-y)}{(c-a)(c-b)} \equiv 1.$

14. $\Sigma \dfrac{bc(a-x)(a-y)}{(a-b)(a-c)} \equiv xy.$

15. $\dfrac{a^2+x^2}{(a-b)(a-c)} + \dfrac{b^2+x^2}{(b-c)(b-a)} + \dfrac{c^2+x^2}{(c-a)(c-b)} \equiv 1.$

16. $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3 \equiv (b+c)^3+3(b+c)^2a+3(b+c)a^2+a^3.$

What third unsymmetrical expression are they both equal to?

17. $\dfrac{a}{\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)} + \dfrac{b}{\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right)} + \dfrac{c}{\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)} \equiv a+b+c.$

18. $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right) - \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right) \equiv 1.$

215. Extension of Factor Theorem. If $f(x) = 0$ when $x = a$ and also when $x = b$, then $x-a$ and $x-b$ are both factors of $f(x)$ at the same time.

We have assumed this theorem in previous work; the need for it is seen by considering the factors of an arithmetical number, e.g. 36. The factors are 2, 4, 6, 9, 12, 18; but 9 and 18 are not simultaneous factors. Now the factor theorem proves that if $f(a) = 0$, and $f(b) = 0$, $x-a$ and $x-b$ are both factors; we proceed to prove that their product $(x-a)(x-b)$ is a factor.

$\because$ $f(a) = 0$; $\quad\therefore$ $x-a$ is a factor.

Let $f(x) \equiv (x-a) \times \phi(x)$.

Put $x = b$, then

$$f(b) \equiv (b-a) \times \phi(b),$$

but $f(b) = 0$, and $b-a \neq 0$;

$\therefore$ $\phi(b) = 0$.

$\therefore$ $x-b$ is a factor of $\phi(x)$.

Hence $(x-a)(x-b)$ is a factor of $f(x)$.

By continuing this process, the more general theorem can be proved:

If $f(a), f(b), f(c) \ldots$ each $= 0$, then the product

$$(x-a)(x-b)(x-c)\ldots \text{ is a factor of } f(x).$$

216. If an equation of the n^{th} degree is satisfied by more than n different values of the variable, then the equation is an identity.

Let $\quad p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$ *

be the equation when reduced to its simplest form, and suppose it is satisfied by the n roots $a, b, c, \ldots n$, and by some other root q.

By the factor theorem

$$p_0x^n + p_1x^{n-1} + \ldots + p_n \equiv p_0(x-a)(x-b)(x-c)\ldots(x-n).$$

Put $x = q$; then the left-hand side vanishes since $x = q$ is a solution of the equation;

$$\therefore \quad p_0(q-a)(q-b)\ldots(q-n) = 0,$$

but none of the factors $q-a$, $q-b$, ... vanish, since $a \ldots n$ are different from q;

$$\therefore \quad p_0 = 0.$$

The equation is now reduced to $\quad p_1x^{n-1} + p_2x^{n-2} \ldots + p_n = 0.$

This is satisfied by more than $n-1$ values of x, and exactly as before, p_1 is shown to be zero. In the same way all the successive coefficients are shown to vanish. Hence the equation is an identity.

217. If two expressions of the n^{th} degree are equal for more than n values of the variable, then the coefficients of like powers on the two sides must be equal; or in other words, the expressions are identically equal.

Let $\quad p_0x^n + p_1x^{n-1} \ldots + p_n = q_0x^n + q_1x^{n-1} + \ldots + q_n$

for more than n values of x.

* $p_0, p_1, \ldots,$ are just as distinct as different letters $p, q, r, s, \ldots$; but they have the common property of being coefficients. When quantities, although different in value, are used in the same connexion or have some common property, it is useful to denote them by the same letter with subscript numbers or sometimes with dashes.

Then the equation

$$(p_0-q_0)x^n+(p_1-q_1)x^{n-1}+\ \dots\ +(p_n-q_n)=0$$

is satisfied by more than n values of x.

Hence by the previous section

$$p_0=q_0,\ p_1=q_1,\ \dots p_n=q_n.$$

This theorem is generally called the Principle of Indeterminate Coefficients.

Note. In Part I, p. 151, we say, 'The equality will be an identity if the coefficients of x on each side are the same'; in this section we have proved the converse, 'If the equality is an identity, the coefficients of like powers on each side must be the same.'

Suppose $A(x-2)+B(x-3)\equiv 3x-5.$

Up till now we have only been justified in saying that the identity will be true if $A+B=3$ and $2A+3B=5$, thus leaving it doubtful whether there are other ways of satisfying the identity. But now we have proved that $A+B$ must equal 3, and $2A+3B$ must equal 5.

Example I. *Factorize*

$$6x^2+xy-15y^2+8x+26y-8.$$

Step I. Factorize the terms of the second degree

$$6x^2+xy-15y^2=(2x-3y)(3x+5y).$$

Step II. Try to find constants a and b so that

$$(2x-3y+a)(3x+5y+b)\equiv 6x^2+xy-15y^2+8x+26y-8.$$

The terms of the second degree cancel, and there is left

$$(3a+2b)x+(5a-3b)y+ab\equiv 8x+26y-8.$$

Hence by the Principle of Indeterminate Coefficients

$$3a+2b=8$$
$$5a-3b=26$$
$$ab=-8.$$

There are no factors unless these equations can be satisfied by the same values of a and b.

By solving the first two equations, it is found that $a=4$, $b=-2$; and as these values satisfy the third equation, the required factors are $(2x-3y+4)(3x+5y-2)$.

It follows that the graph of

$$6x^2+xy-15y^2+8x+26y-8=0$$

consists of the two straight lines which are the graphs of

$$2x-3y+4=0 \text{ and } 3x+5y-2=0;$$

since any values of x and y which satisfy either of these will also satisfy the original equation.

Example II. *Express* $3n^3-4n^2+5n+2$ *in the form* $An(n-1)(n-2)+B(n)(n-1)+Cn+D$.

Since $3n^3-4n^2+5n+2 \equiv A(n^3-3n^2+2n)+B(n^2-n)+Cn+D$, we have by equating coefficients of like powers

$$\begin{aligned} A &= 3 \\ -3A+B &= -4 \\ 2A-B+C &= 5 \\ D &= 2, \end{aligned}$$

whence $A=3,\ B=5,\ C=4,\ D=2.$

Cf. Weeks, §116; Chrystal, ch. VI, §5.

HIGHEST COMMON FACTOR

218. General Theorems.

(i) If F is a factor of an expression A, it is also a factor of mA where m and A are rational integral expressions.

For suppose $A = F \times a,$

then $mA = m \times F \times a;$

$\therefore$ F is a factor of mA.

(ii) If F is a common factor of two expressions A and B, it is a factor of $pA+qB$ where p and q are any rational integral expressions. Let $pA+qB = S$.

For suppose $A = F \times a, \quad B = F \times b,$

then $pA+qB = paF+qbF$

$= (pa+qb)F;$

$\therefore$ F is a factor of $pA+qB$.

Example. *Find the H. C. F. of* $x^3-6x^2+5x+12$ *and* $2x^3+x^2-12x-27$.

Corollary to (ii).— The H.C.F. of A & B is a factor of S, & hence a C.F. of A & S.

Corollary to (iii).– The H.C.F. of A & S is a factor of B, & hence a C.F. of A & B.

Use this theorem, choosing p and q so as to make x^3 vanish.

Any common factor is a factor of

$$(2x^3+x^2-12x-27)-2(x^3-6x^2+5x+12)$$
$$=13x^2-22x-51$$
$$=(13x+17)(x-3).$$

$13x+17$ cannot be a factor since $13x$ is not a factor of x^3, so that the only possible common factor is $x-3$.

On dividing, or by the remainder theorem, this is found to be a factor of both expressions, and is therefore the H. C. F.

(iii) If F is a factor of A and also of $pA+qB$, it must be a factor of B, provided q is not divisible by F or any factor of F.

For suppose $A = aF$,

and $pA+qB = kF$.

Substitute for A in this last identity,

then $paF+qB = kF$.

Transpose $qB = (k-pa)F$,

hence B must be divisible by F.

In the last example $x-3$ is a factor of $pA+qB$; in order to prove that it is a common factor of A and B this theorem shows that we need divide only one of the quantities A or B.

219. The H. C. F. of any two expressions can be found by a method similar to that used in Arithmetic for finding the G. C. M. of two numbers. By the theorems of this article we can prove that the method is correct.

Example I. *Find the H. C. F. of* $4x^3+4x^2-13x+5$ *and* $6x^4-5x^3-13x^2+17x-5$.

(A) $4x^3+4x^2-13x+5$ / $4x^3-6x^2+2x$

$+10x^2-15x+5$ / $+10x^2-15x+5$

(B) $6x^4-5x^3-13x^2+17x-5$ / 2

$12x^4-10x^3-26x^2+34x-10$ / $12x^4+12x^3-39x^2+15x$

(C) $-22x^3+13x^2+19x-10$ / -2

$44x^3-26x^2-38x+20$ / $44x^3+44x^2-143x+55$

(D) -35 | $-70x^2+105x-35$

(E) $2x^2-3x+1$

H. C. F. $= 2x^2-3x+1$.

Corollary to the two ditto wh. have b^n inserted above.–

The H.C.F of A & B is the H.C.F. of A & S.

Denote the various expressions by the letters A, B, C, D as shown.

All common factors of A and B are factors of $2B-3xA$ (Theorem ii), i. e. are factors of C.

$\therefore$ the H. C. F. is a factor of C and is the H. C. F. of C and A, since all common factors of C and A are also factors of B (Theorem iii).

All common factors of A and C are factors of $-2C-11A$, i. e. are factors of D, i. e. the H. C. F. is a factor of D.

-35 is a factor of D but is clearly not a factor of A nor of B.

$\therefore$ we may reject the factor -35, so that the expression E now contains the H. C. F. if there is one.

Hence the H. C. F. is the H. C. F. of A and E (Theorem iii), but on dividing E is a factor of A.

$\therefore$ E is the H. C. F. of A and E, and therefore of A and D, and therefore of A and B.

When this method has been used several times, it can be shortened by using detached coefficients (Part I, § 116).

Instead of choosing the multipliers so that the term of highest dimensions cancels, it is often advisable to make the constant terms cancel. Consider again the previous example.

(A)

x^3	x^2	x	1
4	+ 4	− 13	+ 5
4	− 6	+ 2	
	+ 10	− 15	+ 5
	+ 10	− 15	+ 5

(B)

x^4	x^3	x^2	x	1	
6	− 5	− 13	+ 17	− 5	(B)
	+ 4	+ 4	− 13	+ 5	
x | 6	− 1	− 9	+ 4		(C)
	6	− 1	− 9	+ 4	(D)
	2				
	12	− 2	− 18	+ 8	
	12	+ 12	− 39	+ 15	
-7 |		− 14	+ 21	− 7	(E)
			2	− 3 + 1	(F)

Here the H. C. F. of A and B

= the H. C. F. of A and $A+B$, i. e. of A and C

= the H. C. F. of A and D since x is clearly not a common factor

= the H. C. F. of $2D-3A$ and A

= the H. C. F. of E and A

= the H. C. F. of F and A since -7 is not a common factor.

NOTE. When a quadratic remainder is reached which cannot itself be the required H. C. F., it should be resolved into factors. Then one at least of the factors can be rejected as not being

a possible common factor. Thus, in the example of § 218, $(13x^2-22x-151)$ cannot be the H.C.F. since 13 is not a factor of 1; on resolving into factors $(13x+17)$ is rejected and only $(x-3)$ remains to be considered.

220. Square Root.

Method I has been given in Chap. XII. p. 126.

Method II. Finding the square root is a particular case of resolving into factors, and if the method of Chapter XII cannot be applied that of § 217 can.

Example. *Find the square root of*

$$x^6+8x^5+12x^4-10x^3+28x^2-12x+9.$$

By the method of Chapter XII it is clear that the first two terms are x^3+4x^2 and that the last term is ± 3.

Assume that

$$(x^3+4x^2+ax+b)^2 \equiv x^6+8x^5+12x^4-10x^3+28x^2-12x+9;$$

then $(2a+16)x^4+(8a+2b)x^3+(a^2+8b)x^2+2abx+b^2$

$$\equiv 12x^4-10x^3+28x^2-12x+9.$$

We have therefore to find whether the values of a and b that satisfy $2a+16=12$ and $8a+2b=-10$,

will also satisfy $a^2+8b=28$,

$2ab=-12$,

and $b^2=9$.

The first two equations give $a=-2$, $b=3$; and these satisfy the other three equations.

$\therefore$ the square root is x^3+4x^2-2x+3.

Method III. The following method is often used:

x^3+4x^2-2x+3 square root

$2x^3$ | $x^6+8x^5+12x^4-10x^3+28x^2-12x+9$

x^6

$2x^3+4x^2$ | $8x^5+12x^4$

$8x^5+16x^4$

$2x^3+8x^2-2x$ | $-4x^4-10x^3+28x^2$

$-4x^4-16x^3+4x^2$

$2x^3+8x^2-4x+3$ | $6x^3+24x^2-12x+9$

$6x^3+24x^2-12x+9$

Square root $= x^3+4x^2-2x+3$.

Step I. Arrange expression in descending powers of x. Take the square root of the term of highest degree and write the square root down at the side or on the top as is done in this example. Subtract the square of the first term from the original first term and bring down the next two terms.

Step II. At the left-hand side place twice the term of the root already found (viz. $2x^3$). Treat this as a divisor and divide the highest term left in the right-hand expression (viz. $8x^5$) by $2x^3$. This gives $4x^2$ as the second term in the root. Before multiplying add this second term to the trial divisor, thus obtaining $2x^3+4x^2$. Multiply, subtract, and bring down the next two terms.

Step III. Add the term of the root last obtained to the last divisor (i.e. add $4x^2$ to $2x^3+4x^2$). The new divisor thus obtained should always be twice the portion of the square root already found. Obtain the next term as in Step II, add this next term $(-2x)$ to the divisor, multiply, and subtract: continue the process till there is no remainder or until a remainder is reached of lower degree than the divisor.

Proof of Method III.

Since $(a+b)^2 = a^2+2ab+b^2$, we know that

$$\sqrt{a^2+2ab+b^2} = a+b.$$

Now we can always find the first term of the square root; subtract the square of the first term of the root from the original expression

$$(a^2+2ab+b^2)-a^2 = 2ab+b^2.$$

In this case we know that the second term of the root is b; but we can get b by dividing the first term of the remainder, viz. $2ab$ by $2a$, i.e. by twice the first term of the root. Now a may stand for any expression, e.g. the first term, or the first two terms or the first three terms of the required root; hence we have the general rule :—

When any portion a of a square root has been found, and the square of that portion has been subtracted from the original expression, the next term in the root can be found by dividing the remainder by twice the portion of the root already found.

The work is arranged as already shown, viz.

$$\begin{array}{ll} & a^2+2ab+b^2 \;\big|\, a+b \\ & \underline{a^2} \\ 2a+b & \quad 2ab+b^2 \\ & \quad \underline{2ab+b^2} \end{array}$$

221. Cube Root.

If a cube root is exact it can usually be found by inspection, or by indeterminate coefficients.

Example I. *Find the cube root of*

$$x^9-3x^8+6x^7-10x^6+12x^5-12x^4+10x^3-6x^2+3x-1.$$

By inspection, the cube root begins with x^3 and ends with -1.
Assume that x^3+px^2+qx-1 is the cube root.

$$\begin{aligned}(x^3+px^2+qx-1)^3 &= \{(x^3+px^2)+(qx-1)\}^3\\ &= (x^3+px^2)^3+\ldots+(qx-1)^3\\ &= x^9+3px^8+\ldots-3qx-1.\end{aligned}$$

Equating the coefficients of x^8 and of x, we see that the only possible exact cube root is x^3-x^2-x-1.
This should now be cubed to make sure that the original expression is an exact cube.

There is a method similar to Method III for square roots. In this the trial divisor is three times the square of the portion of the root already found, to this we add three times the product of the portion already found and the new term, together with the square of the new term.

Example II. *Find the cube root of*

$$a^6+6a^5+15a^4+20a^3+15a^2+6a+1.$$

a^2+2a+1 cube root.

$a^6+6a^5+15a^4+20a^3+15a^2+6a+1$

a^6

$3a^4+6a^3+4a^2$	$6a^5+15a^4+20a^3$
$3a^4+12a^3+12a^2$	$6a^5+12a^4+8a^3$
$3a^4+12a^3+15a^2+6a+1$	$3a^4+12a^3+15a^2+6a+1$

Cube root a^2+2a+1.

Examples XXIII c.

1. Find the factors of $6x^2-7xy-3y^2-8x+y+2$.

2. Show that the graph of $6x^2+7xy-3y^2-4x+5y-2=0$ consists of two straight lines. Draw the graph.

3. Find the value of c in order that $2x^2+xy-6y^2+3x-4y+c$ may be resolved into factors.

4. Find the values of b and c in order that there shall be no remainder when $x^4+6x^3-20x^2+bx+c$ is divided by x^2-3x+2.

5. Find the H. C. F. and L. C. M. of

$$x^4-2x^3+2x-1 \text{ and } 2x^3-3x^2+1.$$

6. Find the H. C. F. of

$$x^3+2x^2+2x+1 \text{ and } x^4-x^3-x+1.$$

7. Prove that the product of any two algebraical expressions is equal to the product of their H. C. F. and L. C. M.

8. Find the square root of

$$x^6+2x^5+3x^4-8x^3-9x^2-10x+25.$$

9. Find the H. C. F. of

$$4x^4+5x^2+7x+2 \text{ and } 16x^3+10x+7.$$

10. Find the H. C. F. of

$$300-x(1+3x)^2 \text{ and } 192-x(1-3x)^2.$$

11. Find the H. C. F. of

$$x^5-x^3-4x^2-3x-2 \text{ and } 5x^4-3x^3-8x-3.$$

12. Extract the square root of

$$81(x^4+1)+36x(x^2+1)-158x^2.$$

13. Find the square root of

$$\frac{a^2}{b^2c^2}+\frac{b^2}{c^2a^2}+\frac{c^2}{a^2b^2}+2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

14. If m and n are two integers with no common divisor except unity, show that m^2+n^2 and m^2-n^2 can have no common divisor other than 2.

15. Factorize $1+y(1+x)^2(1+xy)$.

16. Find the H. C. F. of

$$9x^4-7x^2+4x-3 \text{ and } 3x^4+2x^3+3x^2+1.$$

17. Factorize $(x+a)^3+(x+b)^3-(2x+a+b)^3$.

18. Find the square root of

$$(x+1)(x+2)(x+3)(x+4)+1.$$

19. Find the square root of

$$x^2(x^2+1)-x(x+1)(2x-1)+\tfrac{1}{4}.$$

20. Find the square root of

$$9x^4+66x^3+163x^2+154x+49.$$

21. Resolve into factors

$$2p^2-4pq+2q^2-3p+3q+1.$$

22. Find the H. C. F. of

$$2x^3-9x^2+11x-3 \text{ and } 4x^3-4x^2-5x+3.$$

23. Find the H. C. F. of

$$9x^5+11x^3-2 \text{ and } 81x^5+11x+4.$$

24. Find the square root of

$$x^4+1-12x(x^2+1)+38x^2.$$

25. Extract the square root of

$$c^4x^2+6c^3x^3+5c^2x^4-12cx^5+4x^6.$$

26. Find the square root of

$$4a^2c^2+9b^2c^2-4a^2c-6abc+12abc^2+a^2.$$

27. Find the H. C. F. of

$$96x^3+52x^2+12x+1 \text{ and } 128x^3+64x^2+14x+1.$$

28. If $2x^4+x^3-6x^2-ax-2$ is exactly divisible by x^2+2x+1, find the value of a.

29. Extract the square root of

$$\frac{1}{9}+\frac{2a}{3}+\frac{7a^2}{9}-\frac{2a^3}{3}+\frac{a^4}{9}.$$

30. Find the square root of

$$x(x+1)(x+2)(x+3)+1.$$

31. Find the value of y which will allow of the following fraction being reduced to lower terms:

$$\frac{x^3+(4y-1)x^2+x+1}{x^3+(4y+1)x^2+3x+1}.$$

32. Find the cube root of

$$1-6a+21a^2-44a^3+63a^4-54a^5+27a^6.$$

33. Find the cube root of

$$x^6-12x^5+54x^4-112x^3+108x^2-48x+8.$$

34. Find the cube root of 27081081027.

35. Divide 1 by $1-x$ to 5 terms. What is the remainder?

36. Find the square root of $1-x$ to 5 terms.

37. Find the cube root of $1-x$ to 3 terms.

38. Find the condition that x^2+px+q and x^2+ax+b should have a common factor.

39. Find the necessary condition that $ax^2+bx+c=0$ and $cx^2+bx+a=0$ should have a common root, if a and c are different quantities.

40. Find numerical values of A, B, C so that

$$\frac{x^2}{(x-2)^3}=\frac{A}{(x-2)^3}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)}$$

may be an identity.

Miscellaneous Equations.

The object of the following examples is to show how the work in connexion with equations may often be lessened; or how equations which appear incapable of solution by elementary methods may be reduced to a solvable form.

222. Quadratic Equations.

Example I. Quadratic equations with literal coefficients are usually solved by factors.

Solve $(a+b)x^2+(a+2b+c)x+(b+c)=0$.

This factorizes at once $\{(a+b)x+(b+c)\}\{x+1\}=0$;

$$\therefore\quad x=-1 \text{ or } -\frac{b+c}{a+b}.$$

Example II. The relations $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$ are very useful and often give a quick solution.

Solve $(x-b)(x-c)=(a-b)(a-c)$.

One root is obviously $x=a$.

The sum of the roots is $b+c$;

$\therefore$ the remaining root is $b+c-a$.

Example III. *Form an equation whose roots exceed the roots of* x^2-5x-8 by 2.

Let $y=x+2$,

then $x=y-2$.

But x satisfies the equation $x^2-5x-8=0$;

$\therefore$ y satisfies the equation $(y-2)^2-5(y-2)-8=0$;

i. e. $y^2-9y+6=0$.

Hence the required equation is $x^2-9x+6=0$.

Example IV. *If x is real, prove that* $\dfrac{3x^2+2}{x^2+2x-1}$ *can have all values except such as lie between 2 and* $-\frac{2}{3}$.

Denote the value of the fraction by k.

Then $$\frac{3x^2+2}{x^2+2x-1} = k.$$

Treat this as a quadratic equation to find x.

Multiplying $3x^2+2 = kx^2+2kx-k.$

Transpose $(3-k)x^2-2kx+2+k = 0.$

But x is real;

$\therefore$ $4k^2-4(3-k)(2+k)$ is positive; (Part I, § 150)

i.e. $k^2-6-k+k^2$ is positive;

i.e. $2k^2-k-6$ is positive;

i.e. $(2k+3)(k-2)$ is positive;

i.e. $2\{k-(-\frac{3}{2})\}\{k-2\}$ is positive;

$\therefore$ k cannot lie between 2 and $-\frac{3}{2}$.

223. Equations of degree higher than second. These can often be reduced to quadratic equations.

Example I. *Solve* $(x-3)(x-4)(x-5)(x-6) = 24.$

Rearrange factors $(x-3)(x-6)(x-4)(x-5) = 24.$

Multiply in pairs $(x^2-9x+18)(x^2-9x+20) = 24.$

Treat x^2-9x as the unknown quantity,

$$(x^2-9x)^2+38(x^2-9x)+336 = 0.$$

Factorize $(x^2-9x+14)(x^2-9x+24) = 0.$

Hence, either $x^2-9x+14 = 0$ or $x^2-9x+24 = 0$, and these can be solved in the usual manner.

A similar method may be used with such equations as

$$2x^2-3x-1+\sqrt{2x^2-3x+5} = 0.$$

Replace $\sqrt{2x^2-3x+5}$ by y so that $2x^2-3x+5 = y^2$.

The equation now becomes $y^2-6+y = 0$;

i.e. $(y+3)(y-2) = 0.$

Hence $\sqrt{2x^2-3x+5} = 2$ or -3.

Example II. *Solve* $\sqrt{2x^2+2x+3}+\sqrt{2x^2+x+3}=2x$.

Notice that $(2x^2+2x+3)-(2x^2+x+3)=x$.

Divide each side of this by the corresponding side of the original equation

$$\sqrt{2x^2+2x+3}-\sqrt{2x^2+x+3}=\tfrac{1}{2}.$$

Add this to the original equation and divide by 2,

$$\sqrt{2x^2+2x+3} = x+\tfrac{1}{4}.$$

Square $$2x^2+2x+3 = x^2+\tfrac{1}{2}x+\tfrac{1}{16},$$

which is an ordinary quadratic.

Example III. *Solve* $(a+bx)(b-ax)=\left(\frac{b}{x}+a\right)\left(b-\frac{a}{x}\right)$.

Remove brackets,

$$ab+b^2x-a^2x-abx^2=\frac{b^2}{x}+ab-\frac{ab}{x^2}-\frac{a^2}{x}.$$

Transpose $$ab\left(x^2-\frac{1}{x^2}\right)=(b^2-a^2)\left(x-\frac{1}{x}\right);$$

$$\therefore \text{ either } x-\frac{1}{x}=0 \text{ or } ab\left(x+\frac{1}{x}\right)=b^2-a^2.$$

These can now both be solved by the ordinary methods.

In equations of this character it is often useful to recollect that

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=\left(x-\frac{1}{x}\right)^2+2.$$

Example IV. *Solve* $x^4-10x^3+26x^2-10x+1=0$.

The peculiarity of this equation is that the coefficient of the highest power is the same as the constant, and the coefficient of the next highest power is the same as the coefficient of the first power. Such equations are called **reciprocal equations**.

Divide by x^2 and rearrange,

$$x^2+2+\frac{1}{x^2}-10\left(x+\frac{1}{x}\right)+24=0;$$

$$\text{i.e. } \left(x+\frac{1}{x}\right)^2-10\left(x+\frac{1}{x}\right)+24=0;$$

$$\therefore \quad x+\frac{1}{x}=6 \text{ or } 4.$$

And the solution can be completed by the ordinary methods for quadratic equations.

224. Simultaneous Quadratic Equations.

Example I. *Solve* $\frac{yz}{y+z} = \frac{6}{5}$, (i)

$$\frac{zx}{z+x} = \frac{3}{4}, \quad \text{(ii)}$$

$$\frac{xy}{x+y} = \frac{2}{3}. \quad \text{(iii)}$$

Invert each equation.

$$\frac{y+z}{yz} = \frac{1}{z} + \frac{1}{y} = \frac{5}{6}. \quad \text{(iv)}$$

Similarly

$$\frac{1}{x} + \frac{1}{z} = \frac{4}{3}, \quad \text{(v)}$$

$$\frac{1}{y} + \frac{1}{x} = \frac{3}{2}. \quad \text{(vi)}$$

Add iv, v, vi $\quad 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{5+8+9}{6}$,

divide by 2 $\quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$.

Subtract iv $\quad \frac{1}{x} = \frac{6}{6}; \quad \therefore \ x = 1.$

Subtract v $\quad \frac{1}{y} = \frac{3}{6}; \quad \therefore \ y = 2.$

Subtract vi $\quad \frac{1}{z} = \frac{2}{6}; \quad \therefore \ z = 3.$

Example II. *Solve* $x(y+z) = a,$ (i)

$$y(x+z) = b, \quad \text{(ii)}$$

$$z(x+y) = c, \quad \text{(iii)}$$

Remove brackets and regard yz, zx, and xy as the unknowns. Add the three equations and divide by 2.

$$yz + zx + xy = \frac{a+b+c}{2}.$$

For convenience put $\quad 2s = a+b+c.$

Then $\quad yz + zx + xy = s.$

Subtract i $\quad yz = s - a.$ (iv)

Similarly $\quad zx = s - b.$ (v)

$$xy = s - c. \quad \text{(vi)}$$

Multiply these last three equations together and take square root.

$$xyz = \pm\sqrt{(s-a)(s-b)(s-c)}.$$

Divide in turn by iv, v, vi

$$x = \pm\sqrt{\frac{(s-a)(s-b)(s-c)}{s-a}},$$

and similar values for y and z; the signs being all + or all −.

225. Let $\alpha, \beta, \gamma \ldots$ be the roots of the equation $f(x) = 0$ where

$$f(x) \equiv p_0x^n + p_1x^{n-1} + \ldots + p_n.$$

Since $f(\alpha), f(\beta), f(\gamma)$, &c., each equal nothing,

$(x-\alpha), (x-\beta), (x-\gamma) \ldots$ are all factors of $f(x)$.

Hence $p_0x^n + p_1x^{n-1} + \ldots + p_n$

$$\equiv p_0(x-\alpha)(x-\beta)(x-\gamma)\ldots$$

$$\equiv p_0\{x^n - (\alpha+\beta+\gamma\ldots)x^{n-1} + (\alpha\beta+\alpha\gamma+\beta\gamma+\ldots)x^{n-2} + \ldots + (-\alpha)(-\beta)(-\gamma)\ldots\}.$$

Therefore by the Principle of Indeterminate Coefficients,

$$\Sigma(\alpha) = -\frac{p_1}{p_0}, \quad \Sigma(\beta\gamma) = \frac{p_2}{p_0}, \ldots$$

$$\alpha\beta\gamma\ldots = +\frac{p_n}{p_0} \text{ if } n \text{ is even}, \quad -\frac{p_n}{p_0} \text{ if } n \text{ is odd};$$

$$\text{i.e.} \quad = (-1)^n\frac{p_n}{p_0}.$$

We have had a particular case of this when dealing with quadratic equations. In Part I, § 149, it has been shown that if α and β are the roots of the quadratic equation

$$ax^2 + bx + c = 0,$$

then

$$\alpha+\beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Questions dealing with symmetric functions of the roots of an equation are most easily dealt with by using these relations.

Example I. *Find the value of* $\Sigma(\alpha^2)$ *and* $\Sigma(\alpha-\beta)^2$, *for each of the equations:*

(i) $8x^2 - 2x + 4 = 0$. (ii) $x^3 - 4x^2 + 3x + 1 = 0$.

Let α and β be the roots of the first equation.

Then $\alpha+\beta = \frac{2}{8}, \quad \alpha\beta = \frac{4}{8}$.

Hence $\Sigma(\alpha^2) = (\alpha+\beta)^2 - 2\alpha\beta = \frac{4}{64} - \frac{8}{8} = -\frac{20}{8}$.

$\Sigma(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = \frac{4}{64} - \frac{16}{8} = -\frac{44}{8}$.

The second equation is of the third degree and may have three roots.

Let α, β, γ be the roots,

then $\alpha+\beta+\gamma=4, \quad \beta\gamma+\gamma\alpha+\alpha\beta=3, \quad \alpha\beta\gamma=-1.$

$$\Sigma(\alpha^2)=(\alpha+\beta+\gamma)^2-2(\beta\gamma+\gamma\alpha+\alpha\beta)=16-6=10$$

$$\begin{aligned}\Sigma(\alpha-\beta)^2 &= \alpha^2-2\alpha\beta+\beta^2+\beta^2-2\beta\gamma+\gamma^2+\gamma^2-2\gamma\alpha+\alpha^2\\ &= 2\{\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta\}\\ &= 2(10-3)\\ &= 14.\end{aligned}$$

Example II. *If α and β are the roots of the equation*

$$ax^2+bx+c=0;$$

form the equation whose roots are $1+\frac{\alpha}{\beta}$ *and* $1+\frac{\beta}{\alpha}$.

Required equation is

$$\left\{x-\left(1+\frac{\alpha}{\beta}\right)\right\}\left\{x-\left(1+\frac{\beta}{\alpha}\right)\right\}=0;$$

i.e. $$x^2-\left(2+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\left(1+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+1\right)=0;$$

but $$2+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+2\alpha\beta+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2}{\alpha\beta}$$

$$=\frac{\frac{b^2}{a^2}}{\frac{c}{a}}=\frac{b^2}{ac}.$$

Hence equation is

$$acx^2-b^2x+b^2=0.$$

226. Elimination.

If there are more equations than there are unknown quantities there must be some relation between the coefficients if they are all satisfied by the same values of the unknown quantities. This relation is found by eliminating the unknown quantities.

Example I. *Find the condition that $ax=b$, $cx=d$, should both be true at the same time.*

Clearly the condition is $\frac{b}{a}=\frac{d}{c}$ i.e. $bc-ad=0$.

Example II. *Eliminate x from*

$$ax^2+bx+c=0,$$

and

$$a'x^2+b'x+c'=0.^*$$

This could be done by solving each equation separately and then equating the two values. This method would be very long, and the following method should always be adopted:

$$ax^2+bx+c=0. \quad \text{(i)}$$

$$a'x^2+b'x+c'=0. \quad \text{(ii)}$$

Eliminate x^2.

Multiply i by a' and ii by a.

$$aa'x^2+a'bx+a'c=0.$$

$$aa'x^2+ab'x+ac'=0.$$

Subtract $$(a'b-ab')x=ac'-a'c. \quad \text{(iii)}$$

Eliminate the constant.

Multiply i by c' and ii by c.

$$ac'x^2+bc'x+cc'=0.$$

$$a'cx^2+b'cx+cc'=0.$$

Subtract $$(ac'-a'c)x^2+(bc'-b'c)x=0.$$

Now x is not zero, therefore we may divide by x;

$$\therefore \quad (ac'-a'c)x=b'c-bc'. \quad \text{(iv)}$$

From iii and iv we get two different forms for the value of x; equating them, we have the result of elimination

$$\frac{ac'-a'c}{a'b-ab'}=\frac{b'c-bc'}{ac'-a'c}.$$

Clearing of fractions this becomes

$$(ab'-a'b)(bc'-b'c)=(ac'-a'c)^2.$$

Note. Since the two original equations are true for the same value of x, the two expressions ax^2+bx+c, $a'x^2+b'x+c'$, have a common factor. Hence all the work could be applied to the following question. *Find the condition that ax^2+bx+c, and $a'x^2+b'x+c'$ have a common factor.*

Example III. *Eliminate x and y from*

$$a_1x+b_1y+c_1=0,$$

$$a_2x+b_2y+c_2=0,$$

$$a_3x+b_3y+c_3=0.$$

Solve two of the equations and substitute the solutions in the third equation.

* See footnote on p. 340.

Examples XXIII d.

Solve:

1. $x^2+x+1=\sqrt{3}(2x+1)$.

2. $y^2-36=6y+2\sqrt{y^2-6y-28}$.

3. $x(x-a)+(a+b)(a-b)=ax$.

4. $(2x-a)(2x-b)-(2x+a)(x+b)+a(x+4b)=0$.

5. $\dfrac{2a}{x-b}-\dfrac{b}{x-a}=1$.

6. $\sqrt{x^2+2x-4}+\sqrt{x^2+2x+1}=5$.

7. $(x+1)^2=5(\sqrt{x^2+2x+2}-1)$.

8. $(x^2-30x)(x^2-30x+200)+8064=0$.

9. $x+(x^2-ax+b^2)^{\frac{1}{2}}=\dfrac{x^2}{a}+b$.

10. $\sqrt{2x^2-8x+7}+\sqrt{x^2-5x+9}=\sqrt{2x^2-6x+7}+\sqrt{x^2-7x+9}$.

11. Find the relations between the coefficients of the equations $ax^2+bx+c=0$ and $lx^2+mx+n=0$, if the roots of one equation are the reciprocals of the roots of the other.

12. Prove that the sum of the cubes of the roots of the equation $x^2-px+q=0$ is p^3-3pq.

13. Form the quadratic equation whose roots α and β are determined by $\alpha+\beta=5$, $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{5}{6}$.

14. Find the value of k that makes the roots of the following equation equal $\quad x^2-2(2+k)x+k^2+44=0$.

15. If the difference of the roots of the equation $x^2+px+q=0$ is 1, prove that $p^2+4q^2=(1+2q)^2$.

16. What is the condition that the equation

$$2ax(ax+nk)+(n^2-2)k^2=0$$

has real roots?

17. Solve $x^2+y^2=29$, $(x-y)(x^3-y^3)=351$.

18. Solve $2\left(x^2+\dfrac{1}{x^2}\right)-9\left(x+\dfrac{1}{x}\right)+14=0$.

19. Solve $\dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{9}{2}$, $\dfrac{x}{y^2}+\dfrac{y}{x^2}=\dfrac{9}{4}$.

20. If α and β are the roots of $ax^2+bx+c=0$, find an equation whose roots are $\frac{1+\alpha}{\beta}$ and $\frac{1+\beta}{\alpha}$.

21. Solve $\frac{x}{2}=\frac{y}{3}=\frac{z}{6}=\frac{1}{2x+3y+6z}$.

22. Prove that if the roots of $ax^2+2bx+c=0$ are imaginary, the roots of $ax^2+2(a+b)x+a+2b+c=0$ are also imaginary.

23. Construct an equation whose roots shall exceed the roots of $ax^2+2bx+c=0$ by three.

24. What is the condition that one of the roots of the equation $ax^2+2bx+c=0$ is the reciprocal of one of the roots of the equation $a'x^2+2b'x+c'=0$?

25. Form an equation whose roots are less by 3 than the roots of $x^3-27x-36=0$.

26. Solve $x^4-10x^3+22x^2+10x+1=0$.

27. Show that, if x is real, the fraction $\frac{x^2-3x-2}{2x-3}$ may have any value by properly choosing x. How could this be shown by a graph?

28. It is required to choose k so that the equation

$$x^2-(3k+4)x+k^2+4=0$$

may have real roots. What restriction is there to the value of k?

29. The coordinates of a point are given by $x=4m$, $y=8m^2$, where m is any real quantity. What is the equation of the locus of the point?

30. Eliminate h and k from the equations:

$$\frac{h}{x+24}=\frac{1}{4},\quad \frac{k}{x+15}=\frac{1}{3},\quad x^2+h^2=k^2.$$

31. Find the value of a if the same values of x and y satisfy the three equations

$$2x+3y+5=0,\quad 3x-ay+1=0,\quad x+y-2=0.$$

32. What is the relation between p, q, and r if

$$p=x+(a-1)y,\quad q=x+(b-1)y,\quad r=x+(c-1)y?$$

33. Solve $yz+zx+xy=a^2-x^2=b^2-y^2=c^2-z^2$.

34. Solve $x+y+z=\frac{1}{2}(x^2+y^2+z^2)=-xyz=2$.

35. Solve $x-a=y-b=z-c=\sqrt[3]{xyz}$.

36. Solve $yz+m(y+z)=a^2-m^2$,
$zx+m(z+x)=b^2-m^2$,
$xy+m(x+y)=c^2-m^2$.

37. Solve $x(y+z)=4\frac{1}{2}$,
$y(z+x)=5$,
$z(x+y)=2$.

38. Solve $yz=6$, $zx=3$, $xy=2$.

CHAPTER XXIV

RATIO AND PROPORTION. VARIATION. GRAPHICAL ILLUSTRATIONS. SLOPE OF A CURVE

Ratio and Proportion.

227. Two quantities may be compared either by finding their difference or by finding how many times one is contained in the other. Thus, if we wish to compare the weights of two parcels, one weighing 5 lb. and the other 10 lb., we may say either that one weighs 5 lb. more than the other, since $10-5=5$, or that one weighs twice as much as the other, since $10\div 5=2$. If the weights were 5 lb. and 7 lb. respectively, we should say that the lighter weight is $\frac{5}{7}$ of the heavier weight, since $5\div 7=\frac{5}{7}$. This could otherwise be expressed by saying that the ratio of the lighter weight to the heavier is as 5 is to 7.

The **ratio of two numbers** x and y is the quotient obtained when x is divided by y; the ratio is written $x:y$ and is read: x is to y.

The **ratio of two concrete quantities** *of the same denomination*, containing x units and y units respectively, expresses how many times the second is contained in the first and is measured by the ratio $x:y$.

It is clear from the definition that the ratio of one quantity to another is the same as the multiple or fraction the first is of the second.

If two ratios are equal, the sign :: (read 'as') is sometimes used instead of =; thus, $a:b::c:d$ (a is to b as c is to d) means $a:b=c:d$.

Examples XXIV a.

What is the ratio of:

1. 3 yds. 1 ft. 3 in. to 5 yds.?
2. 3 lb. 10 oz. to 4 lb. 5 oz.?
3. £5 to 3s.?
4. £5 to £3?
5. £5 to 3 yds.?

Are the following statements true?

6. $3:5=27:45$.
7. 3 tons : 5 tons = 27 : 45.
8. 3 tons : 5 tons :: 27 minutes : 45 minutes.
9. 3 tons : 5 cwt. = 27 shillings : 45 pence.

Find x so that the following statements may be true:

10. $4:7=19:x$.
11. 4 ft. : 7 ft. :: 19s. : x shillings.
12. $3:x=$ £14 : £15.
13. $x:5\frac{1}{2}::13$ ft. : 5 yds.
14. Prove that, if $a:b::c:d$, bc must equal ad.

    State and prove the converse.
15. If $x+5:x=19:12$, prove that

    (i) $x+5:5::19:7$, (ii) $x:5::12:7$.
16. If $a:b=c:d$, prove that

    (i) $b:a=d:c$, (ii) $a:c=b:d$,

    (iii) $c:d=a:b$, (iv) $d:c=b:a$.
17. Find two numbers in the ratio of 4 to 7 whose sum is 297.
18. There are two areas A and B; A contains just as many square yards as B contains square metres. Find the ratio of A to B as a decimal assuming that a metre equals 3·28 ft.
19. Two numbers are in the ratio of 3 to 5; the larger of them increased by 2 is to the square root of twice the smaller number as 16 to 3; find the numbers.
20. Two numbers, whose sum is $2a^2$, are in the ratio of $a+b-\frac{ab}{a+b}$ to $a-b+\frac{ab}{a-b}$. Find the numbers.

228. Ratio and Proportion can be treated independently of fractions; consequently there are several expressions in common use the meanings of which must be known although the need for them no longer exists.

The first term of a ratio is called the **antecedent**, the second term the **consequent**.

The ratio $a^2 : b^2$ is called the **duplicate** ratio of $a : b$.

The ratio $a^3 : b^3$ is called the **triplicate** ratio of $a : b$.

The ratio $\sqrt{a} : \sqrt{b}$ is called the **sub-duplicate** ratio of $a : b$.

&c., &c.

The ratio $a : c$ is said to be **compounded** of the ratios $a : b$ and $b : c$.

Since $a : c = \frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$, we see that ratios can be compounded by multiplying the corresponding fractions.

A ratio is said to be of **greater inequality, equality,** or **less inequality** according as the first term is greater than, equal to, or less than the second.

229. If $a : b = c : d$, then a, b, c, d are said to be **in proportion**; a and d are called **extremes**, b and c **means**; also d is called the **fourth proportional** to a, b, and c.

If $a : b = b : c$, then b is called the **mean proportional** between a and c, while c is called the **third proportional** to a and b.

If $a : b = b : c = c : d \ldots$, then a, b, c, $d, \ldots$ are said to be in **continued proportion**.

$a : b : c :: x : y : z$ is an alternative method of writing

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}.$$

230. If $a : b = c : d$, we have already seen that $\frac{a}{b} = \frac{c}{d}$, so that questions involving proportion can be solved by replacing the ratios by fractions. In this way the following general theorems may be proved:

If $a : b = c : d$, then

$b : a = d : c$, (**invertendo**)

$a : c = b : d$, (**alternando**)

$a+b : b = c+d : d$, (**componendo**)

$a-b : b = c-d : d$, (**dividendo**)

$a+b : a-b = c+d : c-d$, (**componendo et dividendo**),

and $bc = ad$.

These results should be recollected and may always be referred to by the corresponding Latin word.

We give a proof of one of these, and leave the rest as exercises for the student.

$$\frac{a}{b} = \frac{c}{d}.$$

Add 1 to each side.

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\frac{a+b}{b} = \frac{c+d}{d};$$

i. e. $a+b : b = c+d : d.$

231. The ratio of two positive quantities a and b is made more nearly equal to unity by adding the same positive quantity to both of them.

It is required to show that $\frac{a+x}{b+x}$ is more nearly equal to 1 than $\frac{a}{b}$, when a, b, and x are all positive.

Case I. $a > b$; then $a : b$ and $a+x : b+x$ are greater than 1.

$$\frac{a}{b} - \frac{a+x}{b+x} = \frac{(a-b)\,x}{b\,(b+x)}, \text{ which is positive.}$$

$$\therefore \quad \frac{a}{b} > \frac{a+x}{b+x} > 1.$$

Case II. $a < b$; then $a : b$ and $a+x : b+x$ are less than 1.

$$\frac{a+x}{b+x} - \frac{a}{b} = \frac{(b-a)\,x}{b\,(b+x)}, \text{ which is positive.}$$

$$\therefore \quad 1 > \frac{a+x}{b+x} > \frac{a}{b}.$$

These two cases are sometimes stated as separate propositions:—

A ratio of greater inequality is diminished by increasing the antecedent and consequent by the same quantity.

A ratio of less inequality is increased by increasing the antecedent and consequent by the same quantity.

232. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then each fraction equals $\frac{a+c+e+\ldots}{b+d+f+\ldots}$.

Let k denote the value of each fraction;

then
$$a=bk,$$
$$c=dk,$$
$$e=fk,$$
$$\ldots$$

Add
$$(a+c+e+\ldots)=(b+d+f+\ldots)k;$$
$$\therefore\ k=\frac{a+c+e+\ldots}{b+d+f+\ldots}.$$

In a similar way the following general theorem may be proved:

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, each fraction equals
$$\left\{\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right\}^{\frac{1}{n}}.$$

Example I. *If* $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$, *then*
$$\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$$

By inverting
$$\frac{b+c}{x}=\frac{c+a}{y}=\frac{a+b}{z}$$

Each fraction $=\frac{(c+a)+(a+b)-(b+c)}{y+z-x}$
$$=\frac{2a}{y+z-x}.$$

By symmetry each fraction $=\frac{2b}{z+x-y}$ or $\frac{2c}{x+y-z}$;
$$\therefore\ \frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$$

Example II. *Solve* $\frac{(x+a)^2+(x-b)^2}{a^2+b^2}=\frac{(x+a)^2-(x-b)}{2ab}$.

[Each fraction $= \dfrac{[(x+a)^2+(x-b)^2] \pm [(x+a)^2-(x-b)^2]}{(a^2+b^2) \quad \pm \quad 2ab}$, the like signs being taken together.]

By (i) adding, (ii) subtracting the numerators and denominators, each side $= \dfrac{2(x+a)^2}{(a+b)^2} = \dfrac{2(x-b)^2}{(a-b)^2}$.

Take square root $\dfrac{x+a}{a+b} = \pm \dfrac{x-b}{a-b}$.

Case I. $\dfrac{x+a}{a+b} = \dfrac{x-b}{a-b} = \dfrac{a+b}{2b}$ $\left[\text{i.e.} = \dfrac{(x+a)-(x-b)}{(a+b)-(a-b)}\right]$;

$\therefore \quad x = \dfrac{a^2+b^2}{2b}$.

Case II. $\dfrac{x+a}{a+b} = \dfrac{-(x-b)}{a-b} = \dfrac{a+b}{2a}$;

$\therefore \quad x = \dfrac{-a^2+2ab+b^2}{2a}$.

This equation may also be solved as follows:

$$\frac{(x+a)^2+(x-b)^2}{(x+a)^2-(x-b)^2} = \frac{a^2+b^2}{2ab}.$$

Componendo et dividendo:

$$\frac{2(x+a)^2}{2(x-b)^2} = \frac{a^2+2ab+b^2}{a^2-2ab+b^2}.$$

Take square root

$$\frac{x+a}{x-b} = \frac{a+b}{a-b} \quad \text{or} \quad \frac{x+a}{x-b} = \frac{-a-b}{a-b}.$$

Dividendo $\dfrac{x+a}{a+b} = \dfrac{a+b}{2b}$ or $\dfrac{x+a}{a+b} = \dfrac{-a-b}{-2a}$.

$\therefore \quad x = \dfrac{a^2+b^2}{2b}$ or $\dfrac{-a^2+2ab+b^2}{2a}$.

233. If two integers a and b can be found so that $x:y = a:b$, then x and y are said to be **commensurable**. If no such integers can be found x and y are **incommensurable**.

Thus $\quad \dot{3} : \dot{1}\dot{4} = 33 : 14,$

$\sqrt{3} : \sqrt{12} = 1 : 2,$

$\sqrt{3} : \sqrt{6} = 1 : \sqrt{2};$

therefore $\dot{3}$ and $\dot{1}\dot{4}$ are commensurable, so are $\sqrt{3}$ and $\sqrt{12}$; but $\sqrt{3}$ and $\sqrt{6}$ are incommensurable since $\sqrt{2}$ is a surd.

But even though x and y are incommensurable, integers a and b can always be found, so that $x:y = a:b$ to any required degree of accuracy. Hence the results already proved on the understanding that the quantities are commensurable are true also for incommensurable quantities. Thus if work is required to be correct to 4 significant figures, we have $\sqrt{2} = 1{\cdot}414$.

$$\therefore \quad \sqrt{3}:\sqrt{6} = 1:1{\cdot}414 = 500:707,$$

and for the required degree of accuracy we have found the ratio $\sqrt{3}:\sqrt{6}$.

Single quantities are called **incommensurable** when they are incommensurable with unity, e.g. $\sqrt{7}$, π (the ratio of the length of circumference of any circle to that of its diameter), are incommensurable.

Examples XXIV b.

If $a:b = c:d$, prove :

1. $2a:3b = 2c:3d$.

2. $3a-5c:2a+3c = 3b-5d:2b+3d$.

3. $a^2+ab:c^2+ca = ab:cd$.

4. $(a+b)(c+d) = \dfrac{b}{d}(c+d)^2 = \dfrac{d}{b}(a+b)^2$.

5. $\left(\dfrac{1}{a}+\dfrac{1}{d}\right)-\left(\dfrac{1}{b}+\dfrac{1}{c}\right) = \dfrac{(a-b)(a-c)}{abc}$.

6. Find a fourth proportional to $\frac{2}{7}$, $\frac{3}{4}$, $\frac{4}{9}$; and a mean proportional to ·017 and ·153.

7. Prove that

$$x^2-4xy+y^2 : x^2-2\sqrt{3}xy-y^2 = x^2+2\sqrt{3}xy-y^2 : x^2+4xy+y^2.$$

Show that each ratio $= x-(2-\sqrt{3})y : x+(2-\sqrt{3})y$.

8. There are two numbers in the ratio 2:3; if 20 be added to each, the sums are as 6:7. What are the numbers?

9. If $b-4a:b+4a = 1:2$, find the value of $b:a$ and of

$$3b+5a:3b-5a.$$

10. If $a:b = b:c = c:d$, prove that

$$a^3+d^3 : b^3+c^3 = b^3-c^3+d^3 : c^3.$$

11. If $a:b = c:d$, show that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

12. Find a third proportional to (i) $\frac{2}{3}$ and $\frac{4}{7}$, (ii) $\sqrt{2}$ and ·323, (iii) ·323 and $\sqrt{2}$.

13. If five quantities a, b, c, d, e are in continued proportion, show that the ratio of the first to the last is the fourth power of the ratio of any two consecutive quantities.

14. If $\frac{yz-x^2}{1-x} = \frac{zx-y^2}{1-y}$, x and y being unequal, show that each is equal to $\frac{xy-z^2}{1-z}$, and also to $x+y+z$.

15. If $a:b::c:d$, show that
$$2a+3b:2c+3d::3a-4b:3c-4d.$$

16. If $a+b:b+c = c+d:d+a$, prove that
$$a=c \quad \text{or} \quad a+b+c+d=0.$$

17. If a, b, c are positive quantities, show that $\frac{a+c}{b+c}$ is greater or less than $\frac{a}{b}$ according as a is less or greater than b.

If $\frac{a}{b}$ is a positive proper fraction, show that positive integers x and y can always be found such that $\frac{xa+y}{xb+y}$ is equal to any given proper fraction which is greater than $\frac{a}{b}$.

18. If $a:b::c:d$, prove that $\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 2\frac{ac}{bd}$.

19. If x and y are unequal, and $x:y$ is the duplicate ratio of $x-z:y-z$, then z is a mean proportional to x and y.

20. If $x':y':z'::xy:x^2:yz$, prove that
$$x:y:z::x'y':x'^2:y'z'.$$

21. If $\frac{x}{2b+2c-a} = \frac{y}{2c+2a-b} = \frac{z}{2a+2b-c}$, find the value of $a:b$ in terms of x, y, and z.

22. If $a:b::b:c::c:d$, prove that

(i) $a:d::b^3:c^3$;

(ii) $a^3+d^3:b^3+c^3::a^2-bc+d^2:(abcd)^{\frac{1}{2}}$.

23. If $a>b$ and $a>c$, a being positive, and if $a:b::c:d$, prove that $a+d-(b+c)$ is positive.

24. If $\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c}$, prove that

$$\frac{y+z}{a} = \frac{z+x}{b} = \frac{x+y}{c}.$$

Solve the following equations:

25. $\sqrt{x}+\sqrt{b} : \sqrt{x}-\sqrt{b} = a : b.$

26. $x : 27 :: y : 9 :: 2 : x-y.$

27. $\frac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}} = \frac{5x}{a}.$

28. $\frac{x^2+x+1}{x^2-x+1} = \frac{a^2+a+1}{a^2-a+1}.$

29. $\frac{x^3+3x^2-2x+4}{x^3-3x^2+2x+4} = \frac{x^2+3x-2}{x^2-3x+2}.$

30. Find $x : y : z$ from

$$ax+by+cz = 0, \quad a'x+b'y+c'z = 0.$$

Variation.

234. If two changing quantities y and x are so connected that the ratio of simultaneous values of y and x is constant, then y is said to **vary directly as** (or, simply, to **vary as**) x. The symbol $\propto$ signifies 'varies as'.

Hence, if $y \propto x$, then $\frac{y}{x} = k$, where k is a constant (that is, independent of x and y).

Similarly, if $y \propto x^n$, then $y = kx^n$.

If $y \propto \frac{1}{x}$, then y is said to **vary inversely as** x.

Many examples of variation have occurred in Arithmetic under the heading 'Proportion,' 'Rule of Three,' or 'Unitary Method.'

The time a journey takes varies directly as the length of the journey.

The time a journey takes varies inversely as the speed.

The cost of building a wall varies as the length of the wall.

235. If $A \propto B$ when C is constant, and $\propto C$ when B is constant, then $A \propto BC$.

{If the base of a triangle is unchanged, the area varies as the height; if the height is unchanged, the area varies as the base; how does the area change when both height and base change?

The cost of purchasing coals varies as the price per ton if the number of tons consumed is constant, and varies as the number of tons if the price is constant; how does the cost vary if the number of tons and the price both change?}

This extremely important proposition may be proved in two ways.

Method I.

$A \propto B$ when C is constant; $\quad \therefore \quad A = k_1 B,$ (i)

where k_1 does not contain A or B but may contain C.

$A \propto C$ when B is constant; $\quad \therefore \quad A = k_2 C,$ (ii)

where k_2 does not contain A or C but may contain B.

Equations i and ii show that A contains both B and C as factors each to the first degree only;

$$\therefore \quad A = kBC, \tag{iii}$$

where k does not contain A or B or C, i.e. $A \propto BC$.

Method II.

Let A, B, C and a, b, c be two sets of simultaneous values.

Suppose B changes to b while C remains constant, then A changes to some value A', so that A', b, C are simultaneous values.

By definition of variation $\quad \dfrac{A}{A'} = \dfrac{B}{b}.$ (i)

Now while B retains the value b, suppose C changes to c, then A completes the change to a.

Hence $\quad \dfrac{A'}{a} = \dfrac{C}{c}.$ (ii)

Multiplying i and ii $\quad \dfrac{A}{a} = \dfrac{BC}{bc};$

i.e. $\quad \dfrac{A}{BC} = \dfrac{a}{bc} = \text{a constant};$

i.e. $\quad A \propto BC.$

When $A \propto BC$, A is said to vary jointly as BC.

Corollary. If A varies directly as B and inversely as C then A varies as $\frac{B}{C}$.

Example I. *If $x \propto y$ and $y \propto z$, show that $x \propto z$.*

Since $x \propto y$, $x = k_1 y$ where k_1 is constant.

Since $y \propto z$, $y = k_2 z$ where k_2 is constant.

Substituting $x = k_1 k_2 z$ where $k_1 k_2$ is constant;

$$\therefore \quad x \propto z.$$

Example II. *If $T \propto \sqrt{l}$ and if $T = 1$, when $l = 39$, find the value of l when $T = \frac{1}{2}$.*

$$T = k\sqrt{l};$$

$$\text{i.e.} \quad T^2 = k^2 l. \tag{i}$$

But $T = 1$, when $l = 39$;

$$\therefore \quad 1 = k^2 \times 39. \tag{ii}$$

Dividing $$\frac{l}{39} = T^2.$$

Hence when $T = \frac{1}{2}$, $l = \frac{39}{4} = 9\frac{3}{4}$.

Example III. *If $x \propto \frac{abc}{def}$, and x_1, a_1, b_1, c_1, d_1, e_1, f_1 are simultaneous values, find the value x_2 corresponding to the set of values a_2, b_2, c_2, d_2, e_2, f_2.*

$$x = k\frac{abc}{def};$$

$$\therefore \quad x_1 = k\frac{a_1 b_1 c_1}{d_1 e_1 f_1},$$

and $$x_2 = k\frac{a_2 b_2 c_2}{d_2 e_2 f_2}.$$

Divide $$x_2 = x_1 \times \frac{a_2}{a_1} \times \frac{b_2}{b_1} \times \frac{c_2}{c_1} \times \frac{d_1}{d_2} \times \frac{e_1}{e_2} \times \frac{f_1}{f_2}.$$

This is the general question of which the Arithmetic examples of unitary method are particular cases.

Example IV. *Three circular coins of radii $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ of an inch, and of thickness $\frac{1}{10}$, $\frac{1}{10}$, and $\frac{3}{10}$ of an inch respectively, are melted together and formed into a circular coin $\frac{1}{5}$ of an inch in thickness. Assuming that the areas of circles vary as the squares of their radii, find, to three places of decimals, the radius of this single coin.*

If the area of a circle is A square inches and the radius is r inches, then $A \propto r^2$, i.e. $A = kr^2$.

Also the volume of the coin V = Area × thickness.

Hence
$$V_1 = k \times (\tfrac{1}{2})^2 \times \tfrac{1}{20} = \tfrac{1}{80}k,$$
$$V_2 = k \times (\tfrac{1}{4})^2 \times \tfrac{1}{10} = \tfrac{1}{160}k,$$
$$V_3 = k \times (\tfrac{1}{8})^2 \times \tfrac{3}{20} = \tfrac{3}{1280}k.$$

Adding these we obtain the volume of the single coin. Let x inches be required radius.

Then
$$V_1 + V_2 + V_3 = k \times x^2 \times \tfrac{1}{5};$$
i.e. $\quad \tfrac{27}{1280}k = \tfrac{1}{5}kx^2;$
$$\therefore \quad x^2 = \tfrac{27}{256},$$
$$x = \tfrac{5 \cdot 196}{16};$$
$$\therefore \quad \text{radius} = \cdot 325 \text{ inch.}$$

Examples XXIV c.

1. The following values of two quantities x and y are observed. Is it possible that $y \propto x$?

x	1·1	2·3	2·7	3·2
y	1·4	3·0	3·6	4·3

2. If $x = 7$ when $y = 5$, and if $y = 25$ when $x = 35$, can you find the value of x when $y = 3{\cdot}7$?

3. If $x \propto y/z^2$; and if $x = 2$ when $y = 4$ and $z = 8$, find y when $x = 3$ and $z = 27$.

4. If y equals the sum of two quantities, of which one is constant and the other varies inversely as x; and if $x = 2$ when $y = 0$, and $x = 3$ when $y = 1$, find the value of y when $x = 6$.

5. If $x \propto \dfrac{z}{y^2}$ and $z^2 \propto \dfrac{y}{x}$, show that $x \propto \dfrac{1}{y}$.

6. Given that $z \propto x + y$, and $y \propto x^2$; and that when $x = \frac{1}{2}$, the values of y and z are $\frac{1}{3}$ and $\frac{1}{4}$, express z in terms of x.

7. A quantity y is the sum of two quantities, of which one varies as x and the other as $\dfrac{1}{x}$. Given that when $x = 3$, $y = 14$, and that when $x = \frac{1}{3}$, $y = -\frac{22}{3}$, find the two values which x may have when $y = 2$.

8. The sum of the squares of two quantities varies as the sum of the quantities, and, when the quantities are equal but not zero, each of them is equal to 2. Find the values of one when the other is $\frac{1}{2}$.

9. If $y \propto \sqrt{x}$, and if y is 5 when $x = 3$, state the true formula connecting x and y. What is y if x is 9?

10. If $x \propto y^2 - z^2$, when $y \propto t^2$ and $z \propto t^3$; and if, when $t = 3$, $x = 0$, and, when $t = -1$, $x = 8$, find x when $t = 2$.

11. $y = ax^2 + bx^3$. When $x = 1$, $y = 4{\cdot}3$, and when $x = 2$, y is 30; find a and b. What is y when $x = 1{\cdot}5$?

12. If $s \propto t^2$, and if $s = 64$ when $t = 2$, find the value of t when $s = 196$.

13. If $H \propto D^{\frac{1}{2}} v^3$, and if D is 1810 and v is 10 when H is 620, find H if D is 2100 and v is 13. (Use logarithms.)

14. The horse-power of the engines of a ship varies as the cube of the speed; if the horse-power is 2000 at a speed of 10 knots, what is the power when the speed is 15 knots?

15. The electrical resistance of a wire is $R \propto l/d^2$, where l is length and d is diameter; its weight is $W \propto ld^2$. Show that $R \propto W/d^4$. If a pound of wire of diameter $d = 0{\cdot}06$ inch has a resistance of 0·25 ohm, what is the resistance of a pound of wire of the same material, the diameter being 0·01 inch?

16. If y is equal to the sum of two quantities one of which varies as x and the other as $\frac{1}{x}$; and $y = na + b$ when $x = a$, and $y = a + nb$ when $x = b$, find two values of x which make $y = nab + 1$.

17. If the receipts on a railway vary as the excess of speed above 20 miles an hour, while the expenses vary as the square of that excess, find the speed at which the profits will be greatest, if at 40 miles an hour the expenses are just covered.

18. Explain the statement that the volume of a cone 'varies jointly' as its height and the square of the radius of the base. The height of a certain cone is equal to the radius of its base. If the height were increased by an inch, and the radius of the base were unaltered, 462 cubic inches would be added to the volume; or if the radius were decreased by an inch and the height

were unchanged, 902 cubic inches would be taken from the volume. Find the height and volume of the cone.

19. There are two maps, one to the scale of 2 inches to the mile, the other to the scale of half an inch to the mile. The area of an estate on the first map is 1·46 square inches, what is the area of this estate on the second map?

20. Electric lamp filaments of length l, diameter d, being made of the same material and kept at the same temperature by the application of v volts, the candle power is proportional to $l\,d$ and also to v^2d^3/l. There is a 10-candle power lamp whose $l = 3$ and $v = 100$; we wish to make a 20-candle power lamp with $v = 150$; find l for the new lamp.

21. The value of a coin varies directly as the square of the diameter while its thickness remains the same, and directly as its thickness when its diameter remains the same. Two silver coins have their diameters in the ratio of 4 : 3. Find the ratio of their thicknesses, if the value of the first be four times that of the second.

22. The pressure of the wind on a plane surface varies jointly as the area of the surface and the square of the velocity of the wind. It is found that the pressure on a square foot is 1 lb., when the wind is moving at the rate of 15 miles an hour. Hence find the velocity of the wind when the pressure on a wind gauge 1 sq. yd. in area is 16 lb.

23. In any class of turbine if P is power of the waterfall and H the height of the fall, and n the number of revolutions per minute, then it is known that $n \propto H^{1\cdot25}\,P^{-0\cdot5}$. When the power is 100 horse-power and the fall is 6 feet, there are 50 revolutions per minute. Find n for a fall of 20 feet and 75 horse-power.

24. In uniformly accelerated motion, other things being equal, the distance described from rest varies as the square of the final velocity. Under these circumstances two bodies describe distances such that their final velocities are as 4 is to 5. Also the final velocity of a body describing a distance equal to the sum of those distances is two-thirds of the final velocity due to 1230 feet. Find the two distances.

25. A carrier charges 3*d*. each for all parcels not exceeding a certain weight; and on heavier parcels he makes an additional

charge for every 7 lb. above that weight. The charge for half a cwt. is 1*s.* 3*d.*, and the charge for 126 lb. is five times that for a quarter. What is the scale of charges? Verify by a graphical illustration.

26. The duration of a railway journey varies directly as the distance and inversely as the velocity. The velocity varies directly as the square root of the quantity of coal used per mile and inversely as the number of carriages in the train. In a journey of 25 miles in half an hour with 18 carriages, 10 cwt. of coal is required. How much coal will be consumed in a journey of 15 miles in 20 minutes with 20 carriages?

27. Two persons are awarded pensions which vary as the square root of the number of years they have served. One has served 9 years longer than the other and receives a pension greater by £50. If the length of service of the first had exceeded that of the second by $4\frac{1}{2}$ years only, their pensions would have been in the proportion 9 : 8. How long had they served, and what were their respective pensions?

28. The consumption of coal by a locomotive varies as the square of the velocity. When the speed is 16 miles an hour the consumption of coal is two tons per hour. If the price of coal be 10*s.* per ton, and the other expenses of the engine be 11*s.* 3*d.* an hour, find the least cost of a journey of 100 miles.

236. The following geometrical theorems are useful in the graphical treatment of proportion and variation:

Theorem I. If a straight line is drawn parallel to a side of a triangle, it cuts the other two sides proportionally.

Theorem II. If two triangles are equiangular, the ratios of corresponding sides are equal.

Theorem III. If two triangles have one angle of the one equal to an angle of the other and the sides about the equal angles proportional, then the triangles are equiangular.

237. Theorem I leads to the geometrical method of finding third and fourth proportionals.

Example. *Find a fourth proportional to* 3, 5, 7.

Draw $OA = 3$, and produce to B, making $AB = 5$.

Draw OD making any angle with OB, and cut off $OC = 7$.

Join AC. Draw BD parallel to AC to meet OC produced at D.

Then CD represents the required fourth proportional x;

$\because$ AC is parallel to BD, $OA : AB :: OC : CD$ Th. I, § 236.

i.e. $3 : 5 :: 7 : x$

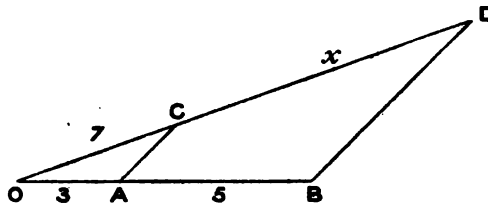

Fig. 58.

238. Theorem II gives the proof of the geometrical method for finding a mean proportional.

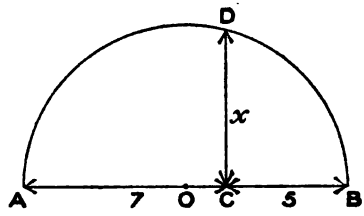

Fig. 54.

Example. *Find a mean proportional between* 5 *and* 7.

Draw $AB = 7+5$ and cut off $AC = 7$.

Bisect AB at O.

On AB describe a semicircle.

Draw CD perpendicular to AB cutting the semicircle at D.

CD represents the required mean proportional x.

For if AD and BD are drawn, ACD and BCD are equiangular triangles; $\therefore$ $AC : CD = CD : BC$,

i.e. $7 : x = x : 5$.

Algebraical Proof.

Take O as origin and AOB as axis of x.

Then equation of circle is $x^2+y^2 = 6^2$,

and equation of CD is $x = 1$,

These meet where $y^2 = 6^2-1^2$;

$\therefore$ $CD^2 = 35 = 5 \times 7$.

i.e. CD is mean proportional between 5×7. The general proof can be written out in exactly the same way.

239. So far the fact that the equation of a straight line is of the first degree has been determined by experience only, we can now give a proof.

To find the equation of a straight line cutting the axes at A and B.

Let $OB = b$, $OA = a$ (where a and b may be positive or negative; in the figure OB is positive, OA negative).

Draw a parallel line through the origin.

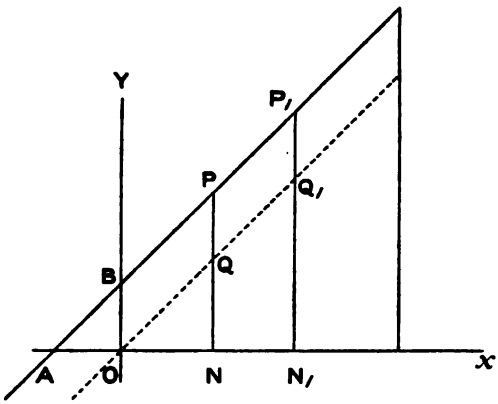

FIG. 55.

Take any number of points $P, P_1, P_2, \ldots$ on the line AB; and let their ordinates $PN, P_1N_1, \ldots$ cut the parallel line at Q and Q_1.

Then the triangles QON, $Q'ON'$ are equiangular;

$$\therefore \quad \frac{QN}{ON} = \frac{Q_1N_1}{ON_1} = \frac{Q_2N_2}{ON_2} = \ldots. \quad \text{Th. II, § 286.}$$

Hence the ratio $QN : ON$ is constant whatever point on the line is taken. Let it equal m.

Then $QN = m \times ON$,

but $QN = PN - PQ = PN - OB = y - b$ and $ON = x$;

$$\therefore \quad y - b = mx, \quad \text{or} \quad y = mx + b.$$

This is true for the coordinates of any point on the line AB and is therefore the equation of AB.

In a similar way it may be shown that the equation of any straight line may be put in the form $y = mx + b$.

By using the converse of Th. II, the converse proposition may be proved, viz. If an equation can be put in the form $y = mx + b$, its graph is a straight line.

240. Proportional Parts (see Part I, § 205).

If the values y_1, y_2 corresponding to x_1 and x_2 are known, it is often required to derive the value y corresponding to an intermediate value x.

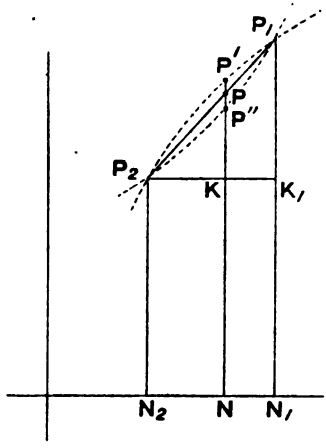

Fig. 56.

The values (x_1, y_1) (x_2, y_2) are represented by the points P_1, P_2 in Fig. 56 and P, P', P'' are possible points corresponding to the value x.

If it is known that P lies on the line P_1P_2, the principle of proportional parts applies accurately, for by equiangular triangles

$$\frac{PK}{P_2K} = \frac{P_1K_1}{P_2K_1},$$

i.e. $$\frac{y-y_2}{x-x_2} = \frac{y_1-y_2}{x_1-x_2}.$$

Th. II, § 236.

And $y-y_2$ can now be found.

But if it is not known that P lies on the line P_1P_2 so that P may be in such a position as P' or P'', then the above proportion is not true.

Usually the values x_1 and x_2 are so close together that the corresponding arc P_1P_2 is almost indistinguishable from a straight line as in Fig. 57, and then the principle of Proportional Parts may be applied with sufficient accuracy for the particular question under discussion.

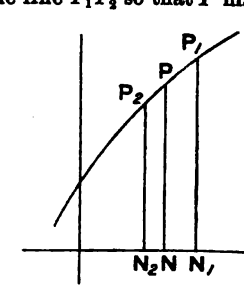

Fig. 57.

Example. *Given* log 3·62 = ·5587, log 3·63 = ·5599, *find* log 3·627.

Here the graph would be practically a straight line;

$$\therefore \quad \frac{\log 3{\cdot}627 - \log 3{\cdot}62}{3{\cdot}627 - 3{\cdot}62} = \frac{\log 3{\cdot}63 - \log 3{\cdot}62}{3{\cdot}63 - 3{\cdot}62},$$

$$\text{i.e.} \quad \text{excess} = \frac{{\cdot}007}{{\cdot}01} \times {\cdot}0012 = {\cdot}0008;$$

$$\therefore \quad \log 3{\cdot}627 = {\cdot}5595.$$

241. Rate of variation. Consider the following sets of values for x, y_1, y_2, y_3:

x	0	1	2	3	4	5
y_1	3	5	7	9	11	13
y_2	0	1	4	9	16	25
y_3	−2	1	4	7	10	13

It is seen that when x increases by 1, y_1 always increases by 2, y_3 always increases by 3, but that the increase of y_2 is at first 1, then 3, then 5, &c. The rate of increase of y_1 is constant and twice the rate of increase of x; y_3 increases three times as fast as x, but the rate of increase of y_2 is variable.

242. If the rate of variation of y is a constant multiple of the rate of variation of x, the corresponding graph is a straight line; but if the rate of variation is not constant, the graph is a curve.

Let x_1, y_1 be a pair of simultaneous values of x and y and let the rate of variation of y be m times that of x.

Then $y - y_1 = m(x - x_1)$.

This is an equation of the first degree if m be a constant.

Therefore the graph is a straight line.

If m be not constant it will be a function of x and y; and the equation $y - y_1 = m(x - x_1)$ will not be of the first degree.

Hence the graph will not be a straight line.

243. In § 239 the constant m which measures the ratio $\frac{QN}{ON}$* depends upon the size of angle QON (which is equal to BAO)

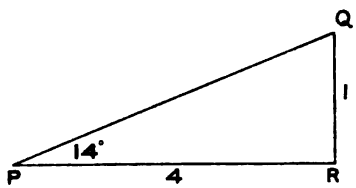

FIG. 58.

and can be used as a measure of the slope of the straight line. Slope may be measured by the angle or by the vertical rise for a

* In Trigonometry this constant ratio is called the tangent of the angle that the straight line makes with the axis of x.

given horizontal distance. Thus in Fig. 58 the slope of PQ is 1 in 4, which is equivalent to saying that PQ makes an angle of 14° with the horizontal.

The slope * or gradient of a straight line is equal to the ratio of the difference of the ordinates of any two points on the line to the difference of the abscissae.

If the equation of a straight line be put in the form $y = mx + b$, then 'm' is the slope or gradient of the straight line.

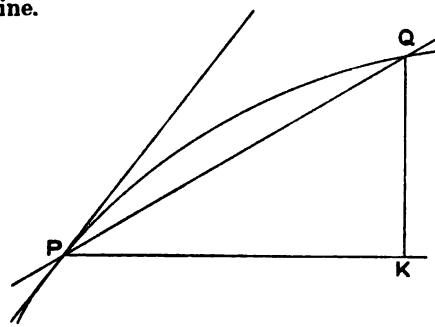

FIG. 59.

244. In the case of a curve the gradient continually changes. The gradient of any chord PQ is called the **average gradient of the curve between the points P and Q.** If Q is indefinitely close to P, so that the chord has become a tangent (Part I, § 151), the gradient of the tangent measures the gradient of the curve at the point P.

Example I. *Verify, that the gradient at any point of the graph of $y = x^2$ is numerically equal to twice the abscissa of the point.*

The curve is shown in Fig. 60, where it must be noticed that the scale for x is twice that for y. This must be allowed for in estimating the gradient.

Tangents are drawn at A, B, C, where $x = 1$, 3, -4, respectively.

Gradient at A is actually $\frac{1}{1}$ which represents $\frac{1}{\frac{1}{2}} = 2 \times 1$.

Gradient at B is actually $\frac{8}{1}$ which represents $\frac{3}{\frac{1}{2}} = 2 \times 3$.

Gradient at C is actually $\frac{4}{-1}$ which represents $\frac{4}{-\frac{1}{2}} = 2 \times (-4)$.

* Some authorities define 'slope' as being the angle itself, but it is more usual to define it as the tangent of the angle.

Note. This figure shows that when the gradient is negative, the angle on the right-hand side of the straight line is obtuse.

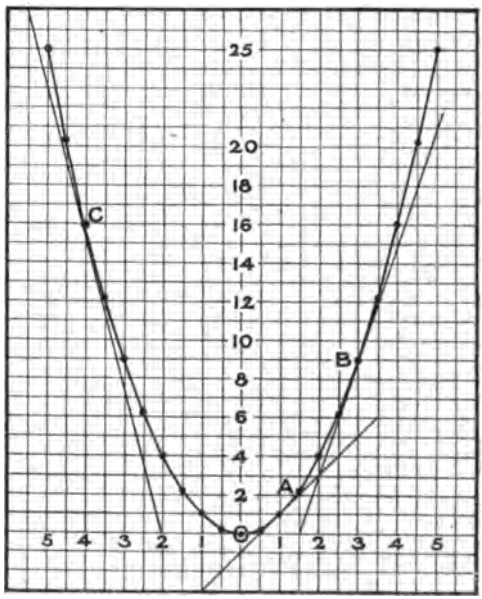

FIG. 60.

245. Since gradient $= \frac{QK}{PK}$, it is seen that the gradient gives the ratio of the rate of increase of y to the rate of increase of x; this leads to many practical applications of graphs and gradients.

If y represents space and x represents time, then the slope represents velocity.

If y represents velocity and x represents time, then the slope represents acceleration.

Examples XXIV d.

1. Find the coordinates of the point which bisects the line joining (2, 3) and (4, 2).

2. Find the coordinates of the points which divide the line joining the points (3, 2) and (7, 9) in the ratio 2 : 3.

3. Find the formula for determining the coordinates of a point which divides the line joining two given points (x_1, y_1), (x_2, y_2) in the ratio $l : m$.

4. Draw the graph of $y = x^2 - 3x + 2$. Determine the slope where x equals (i) -2, (ii) 1, (iii) 3. Plot another graph having the ordinates proportional to the slope of the previous graph. What is the equation of this second graph?

5. Show that the slope at any point on the graph of

$$y = 3x^2 - 2x + 5$$

is numerically equal to $6x - 2$. (Compare with previous example.)

6. A train starting at 12 o'clock on a 25-mile journey passes the points given in the annexed table at the stated times. Plot times horizontally and distances vertically; hence draw a curve showing the speed of the train throughout, and from it find at what distance from the start the speed is greatest, and that speed in miles per hour.

Time	h. m. 12	h. m. 12 4	h. m. 12 8	h. m. 12 13	h. m. 12 20	h. m. 12 27	h. m. 12 31	h. m. 12 38
Distance in miles	0	2	5	10	15	17	20	25

The train stops for 3 minutes at 12 h. 20 m.

7. x and t are the distance in miles and the time in hours of a train from a railway station. Plot on squared paper. Describe why it is that the slope of the curve shows the speed; where approximately is the speed greatest and least?

x	0	1·5	6·0	14·0	19·0	21·0	21·5	21·8	23·0	24·7	26·8
t	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0

8. Values of x and y are given in the following table. By plotting these values, find the ratio of the rate of increase of y to the rate of increase of x when $x = ·47$, ·75, 1 respectively.

x	0	·15	·38	·52	·75	·89	1·20	1·50
y	0	·15	·37	·50	·68	·78	·93	1·00

9. The amount of water in a tank at the end of t seconds from opening a tap is given in the following table:

Time t seconds	0	5	10	15	20	30	40	50	55	60	70
Gallons of water in tank	64	56·2	49	42·3	36	25	16	9	6·25	4	1

Plot these quantities and deduce from the curve the rate at which water was leaving the tank at the end of 15 seconds and 55 seconds.

10. Describe and illustrate by a figure the noticeable features of the curve $y = (x-60)(x-61)(x-62)$.

Find the slope of the tangent to the curve at the point where $x = 61$.

11. The magnetic needle at Greenwich points at present to the west of north. The following were the values in recent years of the angle of deviation from the north:

1898	1899	1900	1901	1902
16° 40′	16° 34′	16° 29′	16° 23′	16° 17′

Find the average rate of decrease of the deviation annually. From these data, find the probable value of the deviation in 1905.

CHAPTER XXV

SERIES. ARITHMETIC, HARMONIC, AND GEOMETRIC PROGRESSIONS. ANNUITIES

Series.

246. **A succession of terms in which the terms are formed according to a fixed law is called a Series.**

Consider the following successions of numbers:

(1) $3+5+7+9+11+\ldots$;

(2) $3+6+12+24+\ldots$;

(3) $3+5+8+13+21+\ldots$;

(4) $3+6+7+11+12+17+21+\ldots$.

In (1) each term after the first is obtained by adding 2 to the preceding term; in (2) each term after the first is obtained by multiplying the preceding term by 2; in (3) each term after the second is obtained by adding the two preceding terms; in

(4) there is no rule for obtaining the terms. Hence (1), (2), and (3) are series but (4) is not a series.

247. In dealing with series we usually try to find (i) the law by which the series is formed, (ii) a formula by which any term can be found without finding the intermediate terms, (iii) a formula by which the sum of any number of consecutive terms may be found without actual addition.

It is useful to denote any term by u with the proper subscript, thus u_5 is the 5th term, u_r is the r^{th} term, &c.; similarly, the sum of n terms is denoted by S_n.

248. Example I. *Discuss the series* $3+7+11+15\ldots.$

Law.

After the first each term is formed by adding 4 to the preceding term. This is usually expressed thus:

The first term is 3 and there is a common difference 4.

Formula for r^{th} term.

The tenth term, for example, is found by adding 4 nine times, i.e. $u_{10} = 3+4\times 9 = 39$.

Similarly, $u_r = 3+4(r-1) = 4r-1$.

This formula should at once be verified by giving r the values 1, 2, 3, 4; this is generally done mentally.

$$u_1 = 4\times 1-1 = 3,\quad u_2 = 4\times 2-1 = 7,\quad u_3 = 4\times 3-1 = 11,$$
$$u_4 = 4\times 4-1 = 15.$$

Hence the formula satisfies all the given conditions.

Sum of n terms.

Whenever the terms proceed by a common difference, the sum can be found as follows:

Consider sum of 18 terms.

$$u_{18} = 4\times 18-1 = 71.$$
$$\therefore\quad S_{18} = 3+7+11+\ldots+63+67+71,$$
$$\text{or}\quad S_{18} = 71+67+63+\ldots+11+7+3;$$
$$\therefore\quad 2S_{18} = 74+74+74+\ldots+74+74+74$$
$$= 18\times 74 \text{ since there are 18 terms,}$$
$$S_{18} = 9\times 74 = 666.$$

S_n can be found by the same method.

Example II. *The r^{th} term of a series is 7.3^{r-1}; write down the first 5 terms and the 29th term and find a formula for the sum of n terms.*

$u_1 = 7.3^{1-1} = 7$, $u_2 = 7.3^{2-1} = 7.3 = 21$, &c., $u_{29} = 7.3^{28}$,

$S = 7 + 7.3 + 7.3^2 + 7.3^3 + 7.3^4 + \ldots$

$= 7 + 21 + 63 + 189 + 567 + \ldots$.

Notice that each term after the first is obtained by multiplying the preceding term by 3. This is usually expressed thus:

The first term is 7 and there is a common ratio 3.

Whenever there is a common ratio, a formula for the sum of n terms can be found.

$$S_n = 7 + 7.3 + 7.3^2 + \ldots + 7.3^{n-2} + 7.3^{n-1}$$

$$3S_n = \quad 7.3 + 7.3^2 + \ldots \quad + 7.3^{n-1} + 7.3^n.$$

Subtract

$2S_n = 7.3^n - 7$ since the intermediate terms cancel;

$$\therefore \quad S_n = \frac{7(3^n - 1)}{2}.$$

Verify by putting $n = 1, 2, 3$.

$$S_1 = \frac{7(3-1)}{2} = 7, \quad S_2 = \frac{7(9-1)}{2} = 28, \quad S_3 = \frac{7(27-1)}{2} = 91.$$

Example III. *Find by actual addition, correct to 3 places of decimals, the sum of 10 terms of the series*

$$1 + \frac{1}{1.2}\left(\frac{1}{3}\right) + \frac{1}{1.2.3}\left(\frac{1}{3}\right)^2 + \frac{1}{1.2.3.4}\left(\frac{1}{3}\right)^3 + \ldots.$$

Here $u_1 = 1$, $u_2 = u_1 \times \frac{1}{2} \times \frac{1}{3}$, $u_3 = u_2 \times \frac{1}{3} \times \frac{1}{4}$,

$u_4 = u_3 \times \frac{1}{4} \times \frac{1}{5}$. Similarly, $u_5 = u_4 \times \frac{1}{5} \times \frac{1}{6}$, &c.

$u_1 = 1$

$u_2 = u_1 \times \frac{1}{2} \times \frac{1}{3} = .166666$

$u_3 = u_2 \times \frac{1}{3} \times \frac{1}{4} = .018518$

$u_4 = u_3 \times \frac{1}{4} \times \frac{1}{5} = .001543$

$u_5 = u_4 \times \frac{1}{5} \times \frac{1}{6} = .000308 \times \frac{1}{6} = .000102$

$u_6 = u_5 \times \frac{1}{6} \times \frac{1}{7} = .000017 \times \frac{1}{7} = .000005$

$\therefore$ Sum correct to 3 decimal places $= 1.187$.

Note. There is no need to calculate any terms beyond u_8 since u_7 begins with 6 noughts, u_8 with 7 noughts, &c., so that their sum cannot affect the third decimal place.

The n^{th} term of this series is $\dfrac{1}{1.2.3.4.\ldots n}\left(\dfrac{1}{3}\right)^{n-1}$.

The product of the first n consecutive integers occurs so often that it has a special symbol, viz. $\lfloor n$, and is called **factorial** n. In some books it is printed $n!$, and is then read n factorial.

Thus the series might be written

$$1+\frac{1}{\lfloor 2}\cdot\frac{1}{3}+\frac{1}{\lfloor 3}\cdot\left(\frac{1}{3}\right)^2+\frac{1}{\lfloor 4}\left(\frac{1}{3}\right)^3+\ldots+\frac{1}{\lfloor n}\left(\frac{1}{3}\right)^{r-1}+\ldots.$$

Examples XXV a.

In each of the following series find the 6^{th} and 10^{th} terms and a formula for u_n. Try to find a formula for S_n.

1. $1+2+3+\ldots.$
2. $1+4+9+\ldots.$
3. $2+4+8+\ldots.$
4. $3+6+12+\ldots.$
5. $3+6+9+\ldots.$
6. $3x+6x^2+12x^3+\ldots.$
7. $1-3+9-27\ldots.$
8. $1-3-7-11\ldots.$
9. $1.2+2.3+3.4+\ldots.$
10. $2+6+12+\ldots.$

Write down the first 5 terms of each of the following series, given that

11. $u_n=2^{n-1}$.
12. $u_n=n(n+1)$.
13. $u_n=7n-2$.
14. $u_r=3^r-r$.
15. $u_n=(-1)^{n+1}\dfrac{3^n}{\lfloor n}$.
16. $u_n=\dfrac{x^n}{n}$.
17. $u_{n+1}=\dfrac{x^n}{\lfloor n}$.

Find the sum of 10 terms of each of the following series and a formula for S_n:

18. $4+6+8+\ldots.$
19. $4+2+0+\ldots.$
20. $a+2a+3a+\ldots.$
21. $a+(a+d)+(a+2d)+\ldots.$
22. $5+10+20+\ldots.$
23. $4+2+1+\ldots.$
24. $a+ar+ar^2+\ldots.$

Find the sum of each of the following series correct to three decimal places, taking as many terms as are necessary:

25. $2+\dfrac{2}{5}+\dfrac{2}{5^2}+\ldots.$
26. $3+\cdot6+\cdot12+\ldots.$

Cf. Chrystal, vol. i, p. 101, ex. 3.

27. $\frac{1}{2}+\frac{1}{2}(\frac{1}{2})^2+\frac{1}{3}(\frac{1}{2})^3+\frac{1}{4}(\frac{1}{2})^4+\ldots.$

28. $1+2x+3x^2+4x^3+\ldots$, when $x=\frac{1}{10}$.

29. Show that

$$a^{n-1}+a^{n-2}x+a^{n-3}x^2+\ldots+a^2x^{n-3}+ax^{n-2}+x^{n-1}=\frac{a^n-x^n}{a-x}.$$

Verify by division.

30. Find, by division, the first 5 terms of a series which is algebraically equivalent to $(1-x)^{-2}$. What is u_n?

249. A series in which the difference obtained by subtracting any term from the following term is constant is called an **Arithmetic Progression.**

Let the first term be denoted by a, the constant difference by d, the n^{th} term by l.

Then the 2nd term $= a+d$,

the 3rd term $= a+d+d = a+2d$,

the 4th term $= a+2d+d = a+3d$,

. . . . .

and the n^{th} term $= a+(n-1)d$,

i. e. $$l = a+(n-1)d. \qquad \text{(i)}$$

In an Arithmetic Progression with first term a, and last term l, the intermediate terms are called **Arithmetic Means** between a and l.

250. To find the sum of n terms of an A. P. (i. e. of an Arithmetic Progression).

$$S_n = a+(a+d)+(a+2d)\ \ldots\ +(l-2d)+(l-d)+l$$

or $$S_n = l+(l-d)+(l-2d)\qquad +(a+2d)+(a+d)+a;$$

$$\therefore\ 2S_n = (a+l)+(a+l)+(a+l)\ \ldots\ +(a+l)+(a+l)+(a+l)$$

$$= (a+l)\times n, \text{ since there are } n \text{ terms};$$

$$\therefore\ S_n = \frac{n}{2}(a+l), \qquad \text{(ii)}$$

but $$l = a+(n-1)d;$$

$$\therefore\ S_n = \frac{n}{2}\{2a+(n-1)d\}, \qquad \text{(iii)}$$

or $$S_n = \frac{n}{2}\{2l-(n-1)d\}. \qquad \text{(iv)}$$

251. In dealing with an Arithmetic Progression there are five quantities to be considered, viz. a, d, n, l, S. The four formulae of §§ 249, 250 each contain four of these letters, and omit one.

$$l = a+(n-1)d \qquad \text{omits } S \tag{i}$$

$$S = \frac{n}{2}(a+l) \qquad \text{omits } d \tag{ii}$$

$$S = \frac{n}{2}(2a+\overline{n-1}d) \quad \text{omits } l \tag{iii}$$

$$S = \frac{n}{2}(2l-\overline{n-1}d) \quad \text{omits } a \tag{iv}$$

It is a useful exercise to work out the missing formula.

252. Example I. *Find the sum of* 50 *terms of an A. P. of which the first term is* 13 *and the last is* 71.

Given $a = 13$, $n = 50$, $l = 71$, find S.

The missing letter is d, therefore use formula (ii).

$$S = \frac{n}{2}(a+l)$$
$$= \frac{50}{2}(13+71) = 50 \times 42$$
$$= 2100.$$

Example II. *The first and last terms of an A. P. are* 3 *and* -37 *respectively, and the sum is* -357, *find the common difference.*

Given $a = 3$, $l = -37$, $S = -357$, find d.

The missing letter is n, and the corresponding formula is so complicated that it is not used. We find n and then find d.

From the formula $S = \frac{n}{2}(a+l)$, we find $n = 21$.

Substituting in the formula $l = a+\overline{n-1}d$, we find $d = -2$.

Example III. *Insert* 5 *arithmetic means between* 3 *and* 14.

In other words, find the intermediate terms of an A.P. of 7 terms, the first and last terms being 3 and 14 respectively.

Here $a = 3$, $l = 14$, $n = 7$, and we must find d.

$$l = a+(n-1)d$$
$$14 = 3+6d$$
$$d = \tfrac{11}{6} = 1\tfrac{5}{6};$$

$\therefore$ required means are $4\frac{5}{6}$, $6\frac{2}{3}$, $8\frac{1}{2}$, $10\frac{1}{3}$, $12\frac{1}{6}$.

Example IV. *A person saves each year £50 more than the previous year: after 10 years his savings amount, without counting interest, to £3000; how much did he save the first year and how much the last year?*

Let £a be his savings in the first year, then in the second year he saves £$(a+50)$, and in the third year £$(a+100)$.

Hence we have to deal with an A. P. in which

$$n = 10, \quad d = 50, \quad S = 3000.$$

To find a, use formula $S = \frac{n}{2}(2a + \overline{n-1}\,d)$.

Substituting $3000 = \frac{10}{2}(2a + 9 \times 50)$

$$10a = 3000 - 2250$$

$$a = 75$$

To find l, use formula $l = a + (n-1)d$.

Substituting $l = 525$.

$$\{\text{Verify by } S = \tfrac{n}{2}(a+l) = 5 \times 600 = 3000\}.$$

In the first year he saved £75; in the last year £525.

Examples XXV b.

Find the sum of:

1. $3\frac{1}{2} + 3\frac{3}{5} + 4 + 4\frac{2}{5} + \ldots$ to 45 terms.
2. $7\frac{1}{2} + 7 + 6\frac{1}{2} + \ldots$ to 35 terms.
3. $9 + 7\frac{1}{2} + 6 + \ldots$ to 12 terms.
4. $3 + 2\frac{3}{5} + 2\frac{1}{5} + \ldots$ to 15 terms.
5. $9\frac{1}{2} + 8\frac{1}{2} + 7\frac{1}{2} + \ldots$ to 25 terms.
6. $2 + \frac{2}{4} + \frac{5}{8} + \ldots$ to 28 terms.
7. $-\frac{1}{4} + \frac{1}{6} + \frac{7}{12} + \ldots$ to 8 terms.
8. $-\frac{1}{5} + \frac{1}{4} + \frac{3}{8} + \ldots$ to 8 terms.
9. $13 + 14{\cdot}2 + 15{\cdot}4 + \ldots$ to 73 terms.
10. $2 + 3 + 5 + 6 + 8 + 9 + \ldots$ to 19 terms and to $2n$ terms.
11. $\frac{1}{2}(n+1) + (n-2) + \frac{3}{2}(n-3) + \ldots$ to $2n-1$ terms.
12. $(4a+b) + (5a) + (6a-b) + \ldots$ to 12 terms.
13. How many terms of the series $3 + 3\frac{1}{8} + 3\frac{1}{4} + \ldots$ must be taken that the sum may amount to $44\frac{1}{4}$? Explain the negative answer.

14. A person pledges his services for a year of 313 working days at the remuneration of 1*d.* for the first day, 2*d.* for the second, 3*d.* for the third, 4*d.* for the fourth and so on. What sum in pounds will he receive in all?

15. What is the sum of an A. P. consisting of 51 terms of which the middle term is 30?

16. Insert 5 Arithmetic means between 5 and 13 and find their sum.

17. *A*'s present wages are 13*s.* a week, and each year he is to receive a rise of 2*s.* a week. *B* starts with the same wages but receives a half-yearly rise of 1*s.* a week. Find the total amounts received by each during the first n years, taking a year as containing exactly 52 weeks.

18. Find the sum of all numbers between 400 and 500 which are divisible by 7.

19. Find the sum of all numbers between 400 and 1000 which are divisible by 17.

20. Find the sum of all the odd numbers between 50 and 4496.

21. The first term of an A. P. is 8, the second term is 7, find the sum of 25 terms.

22. The first term of an A. P. is a, the second term is b, find the sum of n terms. What is the n^{th} term?

23. Two series, $a,\ a+b,\ a+2b, \ldots$

and $a,\ a+c,\ a+2c, \ldots$

are such that the r^{th} term of the first is equal to the $(r^2)^{\text{th}}$ term of the second. Prove that the difference between the sums of n terms of the two series is $\frac{1}{2}rn(n-1)c$.

24. The sum of 10 terms of an Arithmetic Progression is 245, and the ratio of the third term to the sixth is $2:4\frac{1}{2}$. Determine the series.

25. In an arithmetical series, if the common difference be equal to twice the first term, prove that the quotient of the sum of n terms divided by the first term is a perfect square.

26. Prove that the ratio of the sum of x Arithmetic means to the sum of y Arithmetic means inserted between any two numbers is $x:y$.

27. Prove that the sum of $2p+1$ consecutive integers, of which the smallest is p^2+1, is $p^2+(p+1)^2$.

28. The first, second, and last terms of an Arithmetic Progression are respectively a, b, c; show that its sum is

$$\frac{(a+c)(b+c-2a)}{2(b-a)}.$$

253. A series is called a **Harmonic Progression** if the series formed by the reciprocals* of the terms is an Arithmetic Progression.

Thus the series $3+5+7+9+\ldots$ is an Arithmetic Progression; therefore the series $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\ldots$ is a Harmonic Progression.

The terms intermediate between a and l, the first and last terms of a Harmonic Progression, are called the **Harmonic Means** between a and l.

254. To find the sum of n terms of an H. P. (i. e. Harmonic Progression).

There is no formula for finding the sum of an H. P., but in numerical examples the sum of any definite number of terms can be found by actual addition.

All other examples dealing with a Harmonic Progression are solved by means of the corresponding Arithmetic Progression.

Example. *Insert* 15 *Harmonic means between* 3 *and* 47.

First insert 15 Arithmetic means between $\frac{1}{3}$ and $\frac{1}{47}$.

Here $a=\frac{1}{3}$, $n=17$, $l=\frac{1}{47}$, find d.

From the formula $\quad l=a+(n-1)d$

$$\frac{1}{47}=\frac{1}{3}+16d.$$

$$\therefore\quad d=-\frac{44}{16\times 3\times 47}=-\frac{11}{564}.$$

$\therefore$ Arithmetic means are $\frac{117}{564}, \frac{106}{564}, \ldots \frac{28}{564}$,

and required Harmonic means are $\frac{564}{117}, \frac{564}{106}, \ldots \frac{564}{28}$.

* If x is any quantity, then $\frac{1}{x}$ is the reciprocal of x.

Examples XXV c.

1. The first two terms of an H. P. are 1 and 2, what is the 10^{th} term?

2. The 3^{rd} term of an H. P. is 2, the 5^{th} term is 4, what is the 1^{st} term?

3. Find the n^{th} term of the series $\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\ldots$.

4. Find the sum of (i) n terms, (ii) 5 terms of each of the two series:

(i) $1+3+5+\ldots$.

(ii) $1+\frac{1}{3}+\frac{1}{5}+\ldots$.

5. If a, b, c are in H. P. express b in terms of a and c.

6. If a, b, c are in H. P. prove that $a-b : b-c :: a : c$.

7. Insert 3 Harmonic means between 5 and 6.

8. Three quantities T, P, and Q are connected by the relation $T=\frac{2PQ}{P+Q}$, show that they are in H. P. if arranged in the right order.

9. Insert three Harmonic means between $\frac{4}{5}$ and $-2\frac{2}{3}$.

10. Find four numbers in Arithmetic Progression whose sum is 32, the Harmonic mean between the second and third being $7\frac{1}{2}$.

11. Find three numbers in H. P., whose sum is $1\frac{3}{18}$, such that the first is double the last.

12. The first term of an H. P. is a, the n^{th} term is l, find the r^{th} term. Verify your answer as simply as possible.

13. The first term of a Harmonic Progression is 1 and the third is -2. Write down the first five terms of this progression. Find the sum of 13 terms of the Arithmetical Progression, the reciprocals of the terms of which form the above Harmonic Progression.

14. Find x so that $a+x$, $b+x$, $c+x$ may be in Harmonic Progression.

15. The first term of a Harmonic Progression is 5 and the 17^{th} term is $-\frac{11}{4}$. Find the 45^{th} term.

255. A series in which the ratio of any term to the preceding term is constant is called a **Geometric Progression.**

Let the first term be denoted by a, the common ratio by r, and the n^{th} term by l.

Then $\dfrac{2^{\text{nd}}\text{ term}}{1^{\text{st}}\text{ term}} = r$, i.e. second term $= ar$,

similarly the third term $= ar^2$,

and the n^{th} term $= ar^{n-1}$.

Hence $$l = ar^{n-1}. \qquad \text{(i)}$$

The intermediate terms are called **Geometric Means** between a and l.

256. To find the sum of n terms of a G. P. (i.e. Geometric Progression).

$$S = a + ar + ar^2 + \ldots + ar^{n-1}$$
$$rS = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n.$$

Subtract.

If $r > 1$, $(r-1)S = a(r^n - 1)$

$$S = a\frac{r^n - 1}{r - 1}. \qquad \text{(ii)}$$

If $r < 1$, $(1-r)S = a(1-r^n)$

$$S = a\frac{1 - r^n}{1 - r}. \qquad \text{(iii)}$$

257. The formulae of the last two sections enable us to deal with questions involving Geometrical Progressions.

Since the formulae involve r^n, it may be necessary to use logarithms and then the answer cannot, of course, be obtained to more significant figures than are given in the logarithmic tables.

258. Example I. *The second term of a G. P. is* 6, *the sum of the first three terms is* 26, *find the* 5^{th} *term.*

The second term $= ar = 6.$ (i)

Sum of three terms $= a + ar + ar^2 = 26$ (ii)

Divide $$\frac{1 + r + r^2}{r} = \frac{13}{3}.$$

Multiply by $3r$ $\quad 3r^2-10r+3=0$

$$(3r-1)(r-3)=0;$$

$$\therefore \quad r=3, \text{ in which case } a=2;$$

$$\text{or} \quad r=\tfrac{1}{3}, \text{ in which case } a=18.$$

If $a=2$, $r=3$, the 5^{th} term $=2\times 3^4=2\times 81=162$.

If $a=18$, $r=\frac{1}{3}$, the 5^{th} term $=18\times(\frac{1}{3})^4=\frac{2}{3}$.

Example II. *Find the sum of* (i) 6 *terms*, (ii) 27 *terms of the series* $192+96+48+\ldots$.

$\frac{96}{192}=\frac{1}{2}$, $\frac{48}{96}=\frac{1}{2}$; $\quad\therefore$ series is a G. P. in which $a=192$, $r=\frac{1}{2}$.

Since

$r<1$, $S_n=a\dfrac{1-r^n}{1-r}$;

$$\therefore \quad S_6=192\,\frac{1-(\frac{1}{2})^6}{1-\frac{1}{2}}$$

$$=384\,\{1-(\tfrac{1}{2})^6\}$$

$$=384-3\times 2^7\times\frac{1}{2^6}$$

$$=384-6$$

$$=378.$$

$$S_{27}=192\,\frac{1-(\frac{1}{2})^{27}}{1-\frac{1}{2}}$$

$$=384\,\{1-(\tfrac{1}{2})^{27}\}$$

$$=384-3\times 2^7\times\frac{1}{2^{27}}$$

$$=384-\frac{3}{2^{20}}$$

$$=384-\cdot 0000029.$$

$\log\dfrac{3}{2^{20}}=\cdot 4771-20\times\cdot 3010$

$\qquad\qquad=\cdot 4771-6\cdot 020$

$\qquad\qquad=\bar{6}\cdot 4571$ correct to 2 places.

$\therefore \quad \dfrac{3}{2^{20}}=2\cdot 9\times 10^{-6}$.

We notice that the answer $=384$ correct to the 5^{th} decimal place.

Example III. *A man saves* £50 *a year and at the end of each year invests the year's savings at* 3 *per cent. compound interest. How much do his savings amount to at the end of* 20 *years?*

At the end of 1 year, he invests £50.

At the end of 20 years, this £50 has earned interest for 19 years. (Part I, § 205.)

$\therefore$ at the end of 20 years the first year's savings amount to £$50\times(1\cdot 03)^{19}$.

Similarly—

At the end of 20 years, the second year's savings amount to $£50 \times (1{\cdot}03)^{18}$.

At the end of 20 years, the third year's savings amount to $£50 \times (1{\cdot}03)^{17}$.

At the end of 20 years, the 19th year's savings amount to $£50 \times (1{\cdot}03)$.

At the end of 20 years, the 20th year's savings amount to £50.

$\therefore$ at the end of 20 years the total savings

$= 50 + 50 \times 1{\cdot}03 + 50 \times 1{\cdot}03^2 + \ldots + 50 \times (1{\cdot}03)^{19}$ pounds.

$= 50\,\frac{(1{\cdot}03)^{20} - 1}{1{\cdot}03 - 1}$ $\qquad \log (1{\cdot}03)^{20} = 20 \times {\cdot}0128$

$= \frac{5000}{3}\,({\cdot}803)$ $\qquad = {\cdot}256$

$= \frac{4015}{3}$ $\qquad (1{\cdot}03)^{20} = 1{\cdot}803$ approximately

$=$ £1338 approximately. $\qquad = 1{\cdot}803$ correct to 3 figures.

Using four-figure logarithms, we cannot get the answer correct to more than four significant figures. In sums of this kind it is advisable to use seven-figure logarithms.

Examples XXV d.

(Use logarithms where necessary.)

Find the sum of each of the following series:—

1. $2\frac{1}{2} + 1\frac{1}{2} + 1 + \ldots$ to 7 terms.
2. $\frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \ldots$ to 6 terms.
3. $3 + 3\frac{1}{8} + 3\frac{1}{16} + \ldots$ to 10 terms.
4. $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ to 7, 15, and 25 terms.
5. $1 + 2 + 4 + \ldots$ to 7, 15, and 25 terms.
6. $276 + 92 + 30\frac{2}{3} + \ldots$ to 5, 10, and 50 terms.
7. $30\frac{2}{3} + 92 + 276 + \ldots$ to 5, 10, and 50 terms.
8. $2 + {\cdot}2 + {\cdot}02 + {\cdot}002 + \ldots$ to n terms and to 20 terms.
9. $3\frac{1}{2} - 2\frac{1}{3} + 1\frac{1}{6} - 1\frac{1}{21} + \ldots$ to n terms and to 5 terms.
10. $3\frac{1}{2} + 2\frac{1}{3} + 1\frac{1}{6} + 1\frac{1}{21} + \ldots$ to n terms and to 5 terms.
11. $55\,(1 + 1{\cdot}035 + 1{\cdot}035^2 + \ldots)$ to 12 terms.
12. $55\left(1 + \frac{1}{1{\cdot}035} + \frac{1}{1{\cdot}035^2} + \ldots\right)$ to 12, 50, and 100 terms.
13. Insert three Geometric means between 17 and 272.

14. Insert two Arithmetic means and two Geometric means between a^2 and b^2.

15. Insert four Geometric means between $-\frac{1}{10}$ and $3\frac{1}{3}$.

16. Insert one A. mean, one H. mean, and one G. mean between 2 and 8. Arrange them in order of magnitude.

17. Insert one A. mean, one H. mean, and one G. mean between $\frac{1}{3}$ and $\frac{1}{17}$. Arrange them in order of magnitude.

18. Insert one A. mean, one G. mean, and one H. mean between a and b. Show that the product of the A. and H. means equals the square of the G. mean.

19. How many terms of the series 2, -6, 18, &c., must be taken to make -40?

20. Insert two numbers between 5 and 135 so that the four may form a geometrical progression.

21. The population of a town has increased from 160,000 to 212,000 in ten years. What will be its population five years afterwards if it goes on increasing at the same rate?

22. A person is entitled to an annual payment which in each year is less by one-tenth than it was the year before; show that however long it goes on, he cannot receive more than a certain amount in all.

23. What is the condition that the numbers a, b, c should be in Geometric Progression? Apply the test to the numbers 37, 1332, 47752.

24. Find the seventh term of a geometric progression whose first term is 1 and fourth term $-\frac{8}{3\sqrt{3}}$.

25. There is a row of 21 trees planted at intervals of one year. The youngest tree is 10 feet high and the middle tree is 25·9 feet high. Assuming that during each year of growth a tree adds to its height the same fraction of its height at the beginning of that year, find the heights of the oldest tree and of the third tree.

26. Find the sum of n terms of the geometric progression of which the third term is -24 and the sixth term 3. How many terms of the progression must be taken so that the sum may (i) be equal to -64, (ii) may differ from -64 by less than ·001?

27. A person pays to an Insurance Company £23 every year. Reckoning compound interest at 3½ per cent., calculate how much the Company has received at the end of 30 years, a payment having been made at the beginning of the first year.

28. If x, y, z be in G. P., prove that

$$x^2y^2z^2(x^{-3}+y^{-3}+z^{-3}) = x^3+y^3+z^3.$$

29. Prove that the product of $1+x+x^2+\ldots+x^m$ and

$$1-x+x^2\ldots+x^m \text{ is } 1+x^2+x^4\ldots+x^{2m}$$

where m is any even integer.

30. The number of pence in a sum of money is 169 times the sum of the three numbers which appear when it is written in pounds, shillings, and pence. Also those three numbers are in geometrical progression. Determine the common ratio of the progression and write down the sums of money that satisfy the conditions.

259. If $a, x, b,$ are in Arithmetic Progression, x is called the Arithmetic mean between a and b, and is usually denoted by A. Similarly the Geometric and Harmonic means are denoted by G and H.

Since a, A, b are in A. P.,

$$A-a = b-A;$$

$$\therefore \quad A = \frac{a+b}{2}.$$

Since a, G, b are in G. P.,

$$\frac{G}{a} = \frac{b}{G};$$

$$\therefore \quad G^2 = ab,$$

$$\text{i.e.} \quad G = \pm\sqrt{ab}.^*$$

Since a, H, b are in H. P.;

$$\frac{1}{a}, \frac{1}{H}, \frac{1}{b} \text{ are in A. P.;}$$

$$\therefore \quad \frac{2}{H} = \frac{1}{a}+\frac{1}{b},$$

$$\text{i.e.} \quad H = \frac{2ab}{a+b}.$$

* The sign + is usually taken if a and b are positive, − if a and b are negative.

260. Relations between A, G, and H.

(i) $AH = G^2$, i. e. G is the geometric mean between A and H.

This follows at once from the results of the last section.

(ii) $A > G > H$, if a and b are both positive.

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a})^2 - 2\sqrt{ab} + (\sqrt{b})^2}{2} = \frac{(\sqrt{a} - \sqrt{b})^2}{2},$$

but the square of a real quantity is positive;

$\therefore$ $A - G$ is positive, i. e. $A > G$, if a and b are positive.

But $AH = G^2$; $\therefore$ $H < G$ since $A > G$.

Hence $A > G > H$, if a and b are positive.

261. Three quantities a, b, c are in A. P., G. P., or H. P. according as

$$\begin{aligned} a-b : b-c &:: a : a \\ \text{or} \quad &:: a : b \\ \text{or} \quad &:: a : c. \end{aligned}$$

If $a-b : b-c :: a : a$, then $a - b = b - c$;

and a, b, c, are therefore in A. P.

If $a-b : b-c :: a : b$, then $b(a-b) = a(b-c)$,

i. e. $ab - b^2 = ab - ac$,

i. e. $b^2 = ac$;

and a, b, c, are in G. P.

If $a-b : b-c :: a : c$, then $c(a-b) = a(b-c)$,

i. e. $ac - bc = ab - ac$,

i. e. $b(a+c) = 2ac$,

i. e. $b = \frac{2ac}{a+c}$;

and a, b, c are in H. P.

262. **Sum to infinity.** If there is no limit to the number of terms of a series, it is called an **infinite series.**

If the terms continually increase in magnitude, they at length become greater than any assignable number, that

is, the terms become infinite, and in most cases the sum of the infinite series is also infinite. This is the case with all arithmetic progressions and with all geometric progressions with a common ratio greater than unity.

If, however, the terms continually decrease, they at length become smaller than any assigned fraction; that is, the terms become infinitely small. In such a series there may be a limit which the sum of the series cannot exceed, no matter how many terms are taken. This happens in the case of geometrical progressions with a common ratio less than unity.

263. Example I. *Consider the sum of the infinite series*

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+ \dots \text{ to infinity.}$$

This is a G. P. with $a=\frac{1}{2}$, $r=\frac{1}{2}$,

$$S_n=\tfrac{1}{2}\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}=1-(\tfrac{1}{2})^n.$$

As n increases $(\frac{1}{2})^n$ is always positive, hence S_n is always less than 1. But $(\frac{1}{2})^n$ becomes smaller and smaller as n increases and may be made as small as we please by taking n large enough.

Hence S_n is never exactly equal to 1 but may be made to differ from 1 by as little as we please, by taking n large enough. We express this thus:—

The sum to infinity of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \dots$ is 1.

Graphical illustration.

Let OX represent unity.

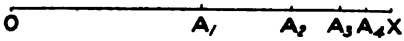

FIG. 61.

Bisect OX at A_1, A_1X at A_2, A_2X at A_3, and so on.

Then $OA_1=\frac{1}{2}$, $A_1A_2=\frac{1}{4}$, $A_2A_3=\frac{1}{8}$, $A_3A_4=\frac{1}{16}$, &c.

Hence OA_2 represents sum of two terms of the series.

OA_3 represents sum of three terms of the series.

OA_4 represents sum of four terms of the series.

and it is clear that the sum of the series is less than OX however many terms are taken, since A_n is always half-way between A_{n-1} and X; but that OA_n can be made as nearly equal to OX as we please by taking n large enough.

Example II. *Prove that the sums of the two progressions*

$$1,\ \tfrac{1}{3},\ (\tfrac{1}{3})^2,\ (\tfrac{1}{3})^3,\ \ldots$$

$$\tfrac{5}{6},\ \tfrac{5}{12},\ \tfrac{5}{24},\ \tfrac{5}{48},\ \ldots$$

approach each other without limit as the number of terms is increased; and find at what place of decimals they differ when 32 terms are taken.

In the first series $a = 1$, $r = \frac{1}{3}$;

$$\therefore \quad S_n = \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} = 5 - 5 \times (\tfrac{1}{3})^n.$$

As n increases without limit, $(\frac{1}{3})^n$ becomes indefinitely small, hence S_n approaches 5 without limit.

In the second series $a = \frac{5}{6}$, $r = \frac{1}{2}$;

$$\therefore \quad S_n = \frac{\frac{5}{6}-\frac{5}{6}(\frac{1}{2})^n}{1-\frac{1}{2}} = 5 - 5 \times (\tfrac{1}{2})^n.$$

As n increases without limit, $(\frac{1}{2})^n$ becomes indefinitely small, hence S_n approaches 5 without limit.

Since the sums of both series approach the same limiting value, they approach each other without limit.

When $n = 32$,

for the first series

$S_n = 5 - 5\,(\cdot 8)^{32}$

$= 5 - \cdot 004$;

$\log (\cdot 8)^{32} = 32 \times \bar{1}\cdot 9031$

$= \bar{4}\cdot 9$

$\cdot 8^{32} = \cdot 0008$

for the second series

$S_n = 5 - 5\,(\cdot 5)^{32}$

$= 5 - \cdot 000000001$,

$\log (\cdot 5)^{32} = \bar{1}\cdot 6990 \times 32$

$= \overline{10}\cdot 3$

$\therefore$ when $n = 32$, the two sums differ at the 3rd decimal place.

264. General formula for sum to infinity of a G. P. when $r<1$.

$$S_n = a\,\frac{1-r^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Since $r<1$, r^n can be made less than any assignable quantity by taking n sufficiently large. Hence S_n can be made to differ from $\frac{a}{1-r}$ by a quantity less than any assignable quantity, i.e. $S_\infty = \frac{a}{1-r}$.

Note. Since r^n continually increases in magnitude as n increases, if r is numerically greater than unity, it is seen that when $r>1$, the sum of an infinite number of terms is ∞.

265. Any sum paid at regular intervals is called an **Annuity.** Unless otherwise stated it is supposed that the annuity is paid yearly.

An annuity may be (1) **terminable,** i.e. payable only for a certain number of years, (2) **perpetual,** i.e. payable for ever, (3) **deferred,** i.e. payable regularly after a certain number of years have elapsed.

Insurance premiums, rent, pensions, &c., are common examples of annuities.

266. If £1 is allowed to accumulate at compound interest at r per cent. for n years, it amounts to $\left(1+\frac{r}{100}\right)^n$.

$1+\frac{r}{100}$ is usually denoted by R.

Hence in n years at r per cent. P pounds amounts to PR^n.

If P denotes the money invested now, and A the sum to which it amounts at the end of n years,

$$A = PR^n.$$

It follows from this formula that the possession of £P now is equivalent to the possession of £A in n years' time, i.e. £P is the **Present Worth** of £A due n years hence.

The formulae for a geometric progression, together with the formula $A = PR^n$, are all that are required to deal with elementary questions on Annuities; no special formulae are needed.

267. Example I. *The rent of a house is £45 per annum, payable yearly. What would be a fair price to pay for a 21 years' lease of the house, interest being reckoned at 4 per cent.?*

Instead of paying £45 at the end of each of 21 years, the purchaser makes a single payment at the beginning of the 21 years.

$$A = PR^n.$$

$r = 4, \quad \therefore\ R = 1{\cdot}04$ and $A = 45$.

The present value of £45 due at end of 1 year is $\frac{45}{1{\cdot}04}$,

The present value of £45 due at end of 2 years is $\frac{45}{(1{\cdot}04)^2}$,

The present value of £45 due at end of 3 years is $\frac{45}{(1{\cdot}04)^3}$.

Total present value

$$= 45\left\{\frac{1}{1{\cdot}04} + \frac{1}{(1{\cdot}04)^2} + \frac{1}{(1{\cdot}04)^3} \ldots 21 \text{ terms}\right\}$$

$$= \frac{45}{1{\cdot}04}\,\frac{1-\left(\frac{1}{1{\cdot}04}\right)^{21}}{1-\frac{1}{1{\cdot}04}}$$

$$= 1125\left[1-\left(\frac{1}{1{\cdot}04}\right)^{21}\right]$$

$$= 1125\,(1-{\cdot}4395)$$

$$= 1125 \times {\cdot}5605$$

$= £630.\ 10s.$ approximately.

$\log(1{\cdot}04)^{-21} = -21 \times {\cdot}0170$
$= -{\cdot}3570$
$= \bar{1}{\cdot}6430$

1125
·5605
562·5
67·5
·5
630·5

Example II. *A man borrows £500 at 5 per cent. compound interest. If he repays by twelve equal annual payments, find the amount of each instalment.*

Let £A be amount of each instalment.

1st Method. **Present Worth Method.**

$$A = PR^n.$$

$\therefore$ Present Worth of 1st payment

$$= \frac{A}{R}.$$

Present Worth of 2nd payment

$$= \frac{A}{R^2}, \text{ \&c.}$$

Also $R = 1{\cdot}05$ and $P = 20$.

$\therefore$ P. W. of all payments

$$= \frac{A}{R}\left(1+\frac{1}{R}+\frac{1}{R^2}+\ldots \text{ 12 terms}\right)$$

$$= \frac{A}{R-1}\left[1-\left(\frac{1}{R}\right)^{12}\right]$$

$$= 20A\left\{1-(1.05)^{-12}\right\}$$

$$= 20A\,(1-.5567)$$

$$= 20A \times .4433$$

$$= 8.866A.$$

$\log (1.05)^{-12} = -12 \times .0212$

$= -.2544$

$= \bar{1}.7456$

But this Present Worth should equal the amount of the debt.

$\therefore \quad 8.866A = 500$

$$A = \frac{500}{8.866}$$

$= £56.\ 8s.$

$$8.866\overline{)500}\ \begin{array}{l} 56.4 \\ (-4433) \\ = 56.70 \quad (-53.2) \\ = 3.50 \end{array}$$

2nd Method. **Interest Method.**

If each payment were at once invested at 5 per cent., the accumulation at the end of 12 years should be the same as if the original £500 had been allowed to accumulate for 12 years.

At the end of 12 years, the first payment A would amount to AR^{11}.

At the end of 12 years, the second payment A would amount to AR^{10}, &c.

At the end of 12 years, the total accumulations would be

$$A\,(R^{11}+R^{10}+R^9+\ldots R+1) = A\,\frac{R^{12}-1}{R-1} = 20A\,(1.05^{12}-1).$$

At the end of 12 years, £500 would amount to 500×1.05^{12}.

$\therefore \quad 20A\,(1.05^{12}-1) = 500 \times 1.05^{12}$

$$A = \frac{500}{20(1-1.05^{-12})} = \frac{500}{20(1-.5567)} \text{ as before.}$$

Example III. *Find the present value of freehold property worth £75 per annum, interest being reckoned at 10 per cent.*

As in Example I, total present value

$$= \frac{75}{1.1} + \frac{75}{(1.1)^2} + \frac{75}{(1.1)^3} + \ldots \text{ to infinity}$$

$$= \frac{75}{1.1}\,\frac{1}{1-\frac{1}{1.1}}$$

$= £750.$

This can easily be checked; for the value is clearly that sum of money which, if invested at 10 per cent., would produce a yearly income of £75. Now £10 is yearly income on £100, therefore £75 is yearly income on £750.

Since $\frac{750}{75} = 10$, the value of the freehold equals ten years' purchase.

Example IV. *A man leaves property worth* £150 *a year to his wife for her life; on her death the property is to pass to his nephew for his life. According to the Insurance Tables it is found that the wife may be expected to live* 12 *years, and the nephew* 34 *years. What is the present value of the annuity to the nephew, interest being reckoned at* 7 *per cent.?*

This is an example of a deferred annuity.

The first payment may be expected at the end of 18 years.

The present value of the first payment $= \frac{150}{(1{\cdot}07)^{13}}$.

The present value of the second payment $= \frac{150}{(1{\cdot}07)^{14}}$.

The present value of the last payment $= \frac{150}{(1{\cdot}07)^{34}}$.

$\therefore$ total present value

$$= \frac{150}{1{\cdot}07^{13}} \frac{1-1{\cdot}07^{-22}}{1-1{\cdot}07^{-1}}$$

$$= \frac{150}{{\cdot}07}\left\{(1{\cdot}07)^{-12}-(1{\cdot}07)^{-34}\right\}$$ which can be worked out by logarithms.

Examples XXVe.

Insert A, G, H means between the following pairs of quantities, and in each case verify that $AH = G^2$, and that $A > G > H$:

1. 3 and 12.
2. 5 and 45.
3. $\frac{1}{3}$ and $\frac{1}{48}$.
4. a and $a+1$.
5. x and x^3.
6. -20 and -45.
7. x^2-x and x^2+x.
8. 3 and -3.
9. 5 and 5.

Find the sum to infinity of the following series:

10. $7+\frac{7}{9}+\frac{7}{9}+\ldots$.
11. $2\frac{1}{2}+\frac{5}{8}+\frac{5}{16}+\ldots$.
12. $\cdot7+2{\cdot}1+6{\cdot}3+\ldots$.
13. $1{\cdot}2+{\cdot}8+{\cdot}4+\ldots$.
14. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$.
15. $6-3+1\frac{1}{2}\ldots$.
16. $\frac{2}{2-x}+\frac{2x}{(2-x)^2}+\frac{2x^2}{(2-x)^3}+\ldots$ when $x<1$.

17. $3-\sqrt{3}+1\ldots.$ 18. $x^5+x^4+x^3+\ldots$, when $x>1$.

19. $\dfrac{x+1}{x^2}-\dfrac{1}{x}+\dfrac{1}{x+1}-\ldots.$

20. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots.$ Verify by working the sum to 5 decimal places.

21. Prove that $\dfrac{\sqrt{6}-\sqrt{3}-\sqrt{2}+1}{2(\sqrt{6}+\sqrt{3}+\sqrt{2}+1)}$ is a Geometric mean between $\dfrac{\sqrt{2}-1}{\sqrt{3}+1}$ and $\dfrac{-7+5\sqrt{2}}{10+6\sqrt{3}}$.

22. The difference between two numbers is 3, and the difference between their Arithmetic and Harmonic means is $\frac{9}{14}$. Find the numbers.

23. The difference between two numbers is 72, and the arithmetic mean between them exceeds the geometric mean by 8. Find the numbers.

24. Find the limit of the sum of the series

$$\tfrac{1}{3}+\tfrac{2}{9}+\tfrac{1}{27}+\ldots \text{ to infinity.}$$

25. If a series of numbers be in arithmetic progression, prove that their reciprocals are in harmonic progression; assuming that A, B, C are defined as being in harmonic progression when $A:C::A-B:B-C$.

26. Form the equation whose roots are the arithmetic and harmonic means between the roots of the equation $x^2-5x+10=0$.

27. If S be the sum to infinity of a geometrical progression whose first term is a, find the sum of the first n terms.

28. Prove that the arithmetic mean of two positive quantities is greater than their geometric mean.

29. Express $\cdot\dot{3}$ as an infinite geometric series, and hence show that $\cdot\dot{3}=\frac{1}{3}$.

30. By means of a G. P. express $\cdot 7\dot{9}6\dot{2}$ as a vulgar fraction.

31. A person repays the principal of a debt by annual instalments, paying £120 the first year, and each succeeding year 10 per cent. more than the year before. How much will he have repaid in 10 years?

32. A person borrows £500 and repays by annual payments of £50. If interest is reckoned at 5 per cent., in how many years will the debt be paid off?

33. Find a geometric series whose second term is 6 and sum to infinity 49.

34. A person spends £50 a year more than his income. Allowing compound interest at 3 per cent., find how long it will be before his accumulated debts amount to £3000.

35. A moneylender advances £12000, and agrees to accept payment of his loan and interest at the rate of 6 per cent. per annum compound interest in eight equal annual instalments. What must be the amount of each instalment?

36. A sum of money was put out at compound interest; the first year's interest was £195. 6*s*. 3*d*.; the fourth year's interest was £219. 14*s*. Find the sum and the rate of interest.

37. A deferred annuity of £20 to continue for 12 years cost £145. 16*s*. When will it commence, the rate of interest being 5 per cent. per annum?

38. In what time will £50 per month discharge a debt of £2000, the nominal rate of interest being $4\frac{1}{2}$ per cent. per annum?

39. Find the present worth of a perpetual annuity of £15 payable at the end of the first year, £30 at the end of the second year, £45 at the end of the third year, and so on; compound interest being taken at 4 per cent. per annum.

40. A man buys the lease of a house for 25 years for £1250. If interest is reckoned at 4 per cent. per annum, how much of the rent ought he to lay aside to form a sinking fund which will just amount to his purchase money when the lease expires?

Examples XXV.

(Miscellaneous examples on Progressions.)

Sum the series:

1. $11\frac{1}{4}+9+6\frac{3}{4}+\ldots$ to 10 terms.
2. $16+12+9+\ldots$ to infinity.
3. $2-\frac{1}{3}+\frac{1}{18}-\frac{1}{108}+\ldots$ to infinity.
4. $(p-3q)+(3p-2q)+(5p-q)+\ldots$ to 21 terms.
5. $1+3+7+15+31+\ldots$ to n terms.
6. $-6\frac{1}{3}-5\frac{2}{3}-5-4\frac{1}{3}+\ldots$ to 20 terms.
7. $\frac{a}{r}+\frac{b}{r^2}+\frac{a}{r^3}+\frac{b}{r^4}+\frac{a}{r^5}+\frac{b}{r^6}+\ldots$ to $2n$ terms.

8. $1+2x+3x^2+4x^3+\ldots$ to n terms and, when possible, to infinity. $S_n=(1-x^n)(1-x)^{-2}-nx^n(1-x)^{-1}$. $S_\infty=\frac{1}{1-x}$, if $x<1$.

9. $3-6+12-24+\ldots$ to 17 terms.

10. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ to n terms.

11. $-\cdot444-\cdot321-\cdot198$ to n terms.

12. $(5a+7b)+(8a+5b)+(11a+3b)+\ldots$ to 18 terms.

13. $\frac{2}{3}-\sqrt{\frac{2}{3}}+1-\ldots$ to 8 terms and to infinity.

14. $x^{\frac{1}{2}}-ax+\frac{a^2}{\sqrt{x}}-\ldots$ to n terms.

15. $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{8}+\ldots$ to n terms.

16. $1-1+1-1+1-1\ldots$ to 54, 79, n terms, and to infinity.

17. $\frac{1}{3}+\frac{5}{9}+\frac{7}{9}+\ldots$ to n terms.

18. $\cdot\dot{1}42857\dot{}+\cdot\dot{2}85714\dot{}+\cdot\dot{4}28571\dot{}+\ldots$ to 8 terms.

19. $\cdot\dot{1}42857\dot{}+\cdot\dot{2}85714\dot{}+\cdot\dot{5}71428\dot{}+\ldots$ to 8 terms.

20. $1+5+13+29+\ldots$ to n terms. What is the r^{th} term?

21. Prove that the sum of n terms of the series

$$1,\ 1+r,\ 1+r+r^2,\ 1+r+r^2+r^3,\ \ldots \text{ is } \frac{n-(n+1)r+r^{n+1}}{(1-r)^2}.$$

22. The population of a country increases by the same number of persons each year throughout n years; the rate of increase per annum in the last year is k times that in the first year; find an expression for the total percentage of increase in the n years.

23. Explain the notation of recurring decimals. Express $0\cdot5\dot{7}\dot{8}$ as a series involving geometrical progression, and thence show by summation that its value is $\frac{28}{495}$.

24. The n^{th} term of a series is $3n-1$, whatever whole number n may be; prove that it is an arithmetic progression, and that the sum of the first $2n$ terms is $n(6n+1)$.

25. A parent puts in a box for a child on every birthday a half-crown for every year of its age. How old will the child be when the value of the money in the box is £17?

26. The side wall of a lean-to shed slopes from a height of 5 feet to a height of 20 feet, and 17 upright slips of wood are fastened to the wall at equal intervals, each reaching from the ground to the top of the wall. Find the total length of wood.

27. A boy arranges rows of marbles one against the other so that each row contains one marble less than the preceding. The last row contains one marble only, which forms the apex of a triangle. If the boy has 153 marbles, how many marbles are there in the base of the biggest triangle he can construct?

28. A ladder is 2 feet 6 inches wide at the base, and 1 foot 4 inches wide at the top. If there be 23 rungs, and if the middle rung be equi-distant from the base and top of the ladder, find the total length of wood which has to be used for the rungs.

29. The yearly output of a gold mine decreases every year 13 per cent. of its amount during the previous year. Given that the first year's output is £260,000, and that $(\cdot 87)^{10} = \cdot 24842$ approximately, find

(a) the total output for the first ten years,

(b) the total output for all time.

30. A number of squares are described whose sides are in G. P. Prove that the areas of the squares are also in G. P.

The side of the $(2m)^{\text{th}}$ square is a feet and the side of the $(2n)^{\text{th}}$ square is b feet; find the area of the $(m+n)^{\text{th}}$ square.

31. A man has charge of 23 machines, each of which when started goes on working automatically and can produce 6·5 yards of material per hour. The man starts the first machine at 9 a.m. and the others at intervals of 5 minutes. What will be the total length of material produced at 1 p.m.?

32. A boy in a swing works up the movement until he swings through a distance of 18 feet in his forward movement. He now remains passive and allows the motion to die away. While the motion is dying away, the distance covered in any movement, either backward or forward, is 5 per cent. less than that covered in the preceding movement in the reverse direction. Find the total distance (including the first 18 feet) that the boy goes while the swing is coming to rest.

33. The population of a certain town at the end of any year is found by subtracting eleven times the population at the end of the previous year from ten times the population at the end of the succeeding year; also ten years ago the population was 11000, and eleven years ago it was 10000. Show that the population increases in geometrical progression.

34. A person purchases the reversion of an estate after 12 years for £1000; what rent ought he to receive that he may realize 6 per cent. per annum on his money?

35. A sum of £3500 is left for three children, *A*, *B*, and *C*, in such a manner that at the end of 7, 9, and 12 years, when they will respectively come of age, they are to receive equal sums; required the present value of each share at 4 per cent. per annum compound interest.

CHAPTER XXVI

MATHEMATICAL INDUCTION: Σn^2 AND Σn^3. MISCELLANEOUS SERIES. REVISION EXAMPLES

Mathematical Induction.

268. When the formula for the sum of a series is known, it can often be proved by 'Induction'. The method is best explained by means of examples.

Example I. *Prove that the sum of the series*

$$1+3+5+\ldots \text{ to } n \text{ terms is } n^2.$$

The r^{th} term is $2r-1$.

Assume that given formula is true when $n = r$.

Then sum of r terms $= r^2$,

the $(r+1)^{\text{th}}$ term $= 2(r+1)-1 = 2r+1$;

$\therefore$ sum of $r+1$ terms = sum of r terms + $(r+1)^{\text{th}}$ term

$$= r^2+2r+1$$

$$= (r+1)^2.$$

Hence if the formula is true for $n = r$, it is also true when $n = r+1$.

When $n = 1$, sum $= 1 = 1^2$.

When $n = 2$, sum $= 1+3 = 2^2$, and formula is true;

$\therefore$ formula is true when $n = 3$,

and therefore when $n = 4$,

and therefore when $n = 5$,

and so on.

Therefore the theorem is true for any value of n.

Example II. *Find the sum of the cubes of the first n natural numbers.*

In other words, find the sum of the series

$$1+2^3+3^3+4^3+\ldots+n^3.$$

$$S_1 = 1 \qquad = 1^2,$$
$$S_2 = 1+8 \quad = 9 \quad = (1+2)^2,$$
$$S_3 = 9+27 \quad = 36 \quad = (1+2+3)^2,$$
$$S_4 = 36+64 = 100 = (1+2+3+4)^2.$$

Hence we have reason to expect that

$$S_n = (1+2+3+\ldots+n)^2 = \frac{n^2(n+1)^2}{4}.$$

Assume this to be true when $n = r$,

then $$S_r = \frac{r^2(r+1)^2}{4},$$

$$u_{r+1} = (r+1)^3;$$

$$\therefore \quad S_{r+1} = \frac{r^2(r+1)^2}{4} + (r+1)^3$$

$$= (r+1)^2\left\{\frac{r^2}{4} + r + 1\right\}$$

$$= \frac{(r+1)^2}{4}(r^2+4r+4)$$

$$= \frac{(r+1)^2(r+2)^2}{4};$$

$\therefore$ if formula is true for S_r it is also true for S_{r+1};
but it is true for $n = 1, 2, 3, 4$;
$\therefore$ it is also true for $n = 5$,
and $\therefore$ it is true when $n = 6$,
and $\therefore$ it is true when $n = 7$,
and so on;
theorem is true for any value of n.

269. Mathematical Induction can be used for other examples than summation of series.

Example. *Prove that when n is a positive integer $(3n+1)7^n-1$ is always divisible by 9.*

Denote expression by $f(n)$.

$f(1) = 4 \times 7 - 1 = 27$ and is divisible by 9,

$f(2) = 7 \times 7^2 - 1 = 342$ and is divisible by 9.

Assume that $f(r) = 9x$, where x is an integer.

Now $f(r+1) = (3r+4)\,7^{r+1} - 1$,

and $f(r) = (3r+1)\,7^r - 1$;

$\therefore\ f(r+1) - f(r) = (3r+4)\,7^{r+1} - (3r+1)\,7^r$

$= 7^r\{7(3r+4) - (3r+1)\}$

$= 7^r\{18r+27\}$;

$\therefore\ f(r+1) = f(r) + 7^r \times 9(2r+3)$

$= 9\{x + 7^r(2r+3)\}$;

i.e. $f(r+1)$ is divisible by 9.

Hence if the theorem is true when $n = r$, it is true when $n = r+1$; but it is true when $n = 1$ and 2, and therefore when $n = 3$, &c.

Therefore theorem is true for any value of n.

Examples XXVI a.

Prove the following statements by induction:

1. $a + ar + ar^2 + \ldots$ to n terms $= a\dfrac{r^n - 1}{r-1}$.

2. $a + (a+b) + (a+2b) + \ldots$ to n terms $= \dfrac{n}{2}(2a + \overline{n-1}\,b)$.

3. $\dfrac{a-b}{a+b} + \dfrac{3a-2b}{a+b} + \dfrac{5a-3b}{a+b} + \ldots$ to n terms

$= \dfrac{n}{a+b}\left(na - \dfrac{n+1}{2}b\right)$.

4. $1^3 + 3^3 + 5^3 + \ldots$ to n terms $= \dfrac{n}{3}(4n^2 - 1)$.

5. $1.2 + 2.3 + 3.4 + 4.5 + \ldots$ to n terms $= \dfrac{n(n+1)(n+2)}{3}$.

6. $2^2 + 5^2 + 8^2 + \ldots$ to n terms $= \dfrac{n}{2}(6n^2 + 3n - 1)$.

7. $n^3 + 5n$ is divisible by 6 if n be any positive integer.

8. $(n+1)(n+2)(n+3)\ldots$ to n factors $= 2^n(1.3.5\ldots$ to n factors).

9. $x^n - 1$ is divisible by $x - 1$ when n is any integer.

10. From n different letters $a, b, c, \ldots$ it is possible to make $n(n-1)$ different expressions of the form $a - b$; but only $\dfrac{n(n-1)}{2}$ different expressions of the form ab.

11. The amount of £P invested at compound interest for n years at r per cent. is $P\left(1+\frac{r}{100}\right)^n$.

12. If there are in a plane n straight lines, of which no three are concurrent, and no two are parallel, the number of points of intersection is $\frac{n(n-1)}{2}$.

13. The n^{th} term of the series 2, 4, 7, 11, 16 is $\frac{n^2+n+2}{2}$.

14. In example 12, the straight lines divide the plane into $\frac{n^2+n+2}{2}$ distinct portions.

270. The sum of the r^{th} powers of consecutive numbers can be found by continued application of the following method, when n is a positive integer.

To find $\Sigma(n)$ and $\Sigma(n)^2$.

$$(n+1)^2-n^2 = 2n+1,$$
$$n^2-(n-1)^2 = 2(n-1)+1 \quad \text{(replacing } n \text{ by } n-1)$$
$$(n-1)^2-(n-2)^2 = 2(n-2)+1.$$
$$\cdots\cdots\cdots\cdots$$
$$2^2-1^2 = 2.1+1,$$
$$1^2-0^2 = 2.0+1.$$

Add $\quad (n+1)^2 = 2\Sigma(n)+n+1;$

$$\therefore\ 2\Sigma(n) = n(n+1),$$
$$\Sigma(n) = \frac{n(n+1)}{2}.$$

{This agrees with the result obtained by using the formula for an A. P.}

$$(n+1)^3-n^3 = 3n^2+3n+1,$$
$$n^3-(n-1)^3 = 3(n-1)^2+3(n-1)+1,$$
$$(n-1)^3-(n-2)^3 = 3(n-2)^2+3(n-2)+1.$$
$$\cdots\cdots\cdots\cdots$$
$$2^3-1^3 = 3(1)^2+3(1)+1,$$
$$1^3-0^3 = 3(0)^2+3(0)+1.$$

Add $(n+1)^3 = 3\Sigma(n^2)+3\Sigma(n)+n+1;$

$$\therefore \quad 3\Sigma(n^2) = (n+1)^3-(n+1)-3\cdot\frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}\{2n^2+4n+2-2-3n\}$$

$$= \frac{n+1}{2}\{2n^2+n\};$$

$$\therefore \quad \Sigma(n^2) = \frac{n(n+1)(2n+1)}{6}.$$

In a similar way it can be shown that

$$\Sigma(n^3) = \frac{n^2(n+1)^2}{4}.$$

271. The formulae for $\Sigma(n)$, $\Sigma(n^2)$, $\Sigma(n^3)$ are often used for the summation of series whose n^{th} term can be expressed in the form An^3+Bn^2+Cn+D.

Example. *Find the number of balls in a triangular pile, each side of the base having* 36 *balls.*

The number of balls in the base

$$= (1+2+3+\ldots+36) = \frac{36\,.\,37}{2}.$$

The number of balls in the next layer

$$= (1+2+3+\ldots+35) = \frac{35\,.\,36}{2}.$$

The number of balls at the top

$$= \frac{1\,.\,2}{2}.$$

$\therefore$ Total number $= \frac{1}{2}\{1\,.\,2+2\,.\,3+3\,.\,4+4\,.\,5+\ldots$ to 36 terms$\}$.

$$n^{\text{th}} \text{ term} = \frac{n(n+1)}{2},$$

$$\sum\frac{n(n+1)}{2} = \tfrac{1}{2}\Sigma(n^2)+\tfrac{1}{2}\Sigma(n)$$

$$= \frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}.$$

$$= \frac{n(n+1)(n+2)}{6}.$$

Hence required number $= \dfrac{36\,.\,37\,.\,38}{6}$

$= 8436.$

272. Further examples of summation of series.

Example I. Indeterminate Coefficients.

To find the sum of $1^2+2^2+3^2+\ldots+n^2$.

Assume $\Sigma(n^2) \equiv A+Bn+Cn^2+Dn^3+\ldots,$

where A, B, C, D, ... are constants.

Change n into $n+1$.

$$\Sigma(n+1)^2 \equiv A+B(n+1)+C(n+1)^2+D(n+1)^3+\ldots.$$

Subtract

$$(n+1)^2 \equiv B+C(2n+1)+D(3n^2+3n+1) \\ E+(4n^3+6n^2+4n+1)+\ldots.$$

Equate coefficients.

$$1 = B+ \quad C+ \quad D+ \quad E+\ldots,$$
$$2 = \quad 2C+3D+4E+\ldots,$$
$$1 = \quad 3D+6E+\ldots.$$

All the other coefficients on the right hand must be zero, they do not contain B, C, or D; hence $E=0$, $F=0$, &c.

Solving the above equations $D=\frac{1}{3}$, $C=\frac{1}{2}$, $B=\frac{1}{6}$.

Hence $$1^2+2^2+3^2+\ldots+n^2 = A+\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}.$$

By putting $n=1$, we see that $A=0$.

Hence
$$\Sigma(n^2) = \frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}$$
$$= \frac{n}{6}\{2n^2+3n+1\}$$
$$= \frac{n(n+1)(2n+1)}{6}.$$

Example II. A series whose sum depends on the sum of a Geometrical Progression.

Find the sum of $1+3x+5x^2+7x^3+\ldots$ *to n terms.*

$$S = 1+3x+5x^2+7x^3+\ldots+(2n-1)x^{n-1},$$
$$xS = \quad x+3x^2+5x^3+\ldots+(2n-3)x^{n-1}+(2n-1)x^n,$$
$$(1-x)S = 1+2x+2x^2+2x^3+\ldots+2x^{n-1}-(2n-1)x^n$$
$$= 1+2x\frac{1-x^{n-1}}{1-x}-(2n-1)x^n;$$
$$\therefore \quad S = \frac{1-(2n-1)x^n}{1-x}+\frac{2x(1-x)^{n-1}}{(1-x)^2}.$$

If $x < 1$, x^n and x^{n-1} become indefinitely small when n is ∞, and by more advanced methods it can be shown that the product nx^n is also indefinitely small when $x = \infty$.*

Hence $$S_\infty = \frac{1}{1-x} + \frac{2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}.$$

This can be verified by division.

Example III. A series in which each term consists of 3 factors in Arithmetic Progression and the respective initial factors are in the same Arithmetic Progression.

Find the sum of n terms of the series

$$1.3.5+3.5.7+5.7.9+\ldots$$

$$u_n = (2n-1)(2n+1)(2n+3),$$

$$u_{n-1} = (2n-3)(2n-1)(2n+1).$$

Hence $$(2n-3)u_n = (2n+3)u_{n-1}; \qquad \text{(i)}$$

also $$(2n+5)u_n - (2n-3)u_n \equiv 8u_n.$$

Therefore $$u_n = \frac{(2n+5)u_n - (2n-3)u_n}{8}$$

$$= \frac{(2n+5)u_n - (2n+3)u_{n-1}}{8} \quad \text{using the result (i).}$$

Similarly $$u_{n-1} = \frac{(2n+3)u_{n-1} - (2n+1)u_{n-2}}{8},$$

$$u_{n-2} = \frac{(2n+1)u_{n-2} - (2n-1)u_{n-3}}{8},$$

. . . . . .

$$u_2 = \frac{11u_2 - 9u_2}{8},$$

$$u_2 = \frac{9u_2 - 7u_1}{8},$$

$$u_1 = \frac{7u_1 + u_1}{8}, \qquad \text{(identically).}$$

Add $u_1 + u_2 \ldots + u_n = \frac{(2n+5)u_n + u}{8}$;

i.e. $$S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \frac{15}{8}.$$

A similar method can be applied to such series as

$$\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots.$$

* By considering such fractions as $\frac{10}{2^{10}}$, $\frac{1000}{2^{1000}}$, it is seen that, if $x < 1$, nx^n continually decreases as n increases.

Examples XXVI b.

Find the sum to n terms of the series whose n^{th} terms are:

1. $n(n+1)$.
2. $(n+a)(n+b)$.
3. $n(n+1)(n+2)$.
4. $n^2(3n-2)$.
5. $3n^2-3n+1$.
6. $n^4-(n-1)^4$.

Find by the method of Indeterminate Coefficients the sum to n terms of the following series, and verify by some other method.

7. $1.2+2.3+3.4+\ldots$.
8. $1.2^2+2.3^2+3.4^2+\ldots$.
9. $1.2+3.4+5.6+\ldots$.
10. $1^2+4^2+7^2+10^2+\ldots$.
11. $1.2.3+2.3.4+3.4.5+\ldots$.
12. $1^3+3^3+5^3+\ldots$.

Find the sum to n terms of:

13. $1+2x+3x^2+4x^3+\ldots$.
14. $a+2ar+3ar^2+4ar^3+\ldots$.
15. $a^2+(a+d)a^2+(a+2d)a^3+(a+3d)a^4+\ldots$.
16. $ab+(a+d)br+(a+2d)br^2+(a+3d)br^3+\ldots$.
17. Show by induction or otherwise that the number of balls in a complete square pile of n layers is $\frac{n(n+1)(2n+1)}{6}$.
18. Find the number of balls in a triangular pyramid if there are 17 balls in a side of the bottom layer.

Find the sum to n terms of:

19. $1.4.7+4.7.10+7.10.13+\ldots$.
20. $2.5.8.11+5.8.11.14+8.11.14.17+\ldots$.
21. $1.2.4+2.3.5+\ldots$.
22. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots$.
23. $\frac{1}{a(a+3)}+\frac{1}{(a+3)(a+6)}+\frac{1}{(a+6)(a+9)}+\ldots$.
24. $\frac{1}{a(a+3)}+\frac{1}{(a+1)(a+4)}+\frac{1}{(a+2)(a+5)}+\ldots$.

Examples XXVI.

Revision Examples.

1. Find the coefficient of x^6 in the product
$(1-2x+4x^2-8x^3+16x^4)\times(1+2x+4x^2+8x^3+16x^4)$.

2. Simplify $\dfrac{(x+3)(2x^2-7x-4)-(x+3)(2x+1)}{(x-5)(2x^2-7x-4)+(x-5)(2x+1)}$.

3. Find the value of $\frac{1}{3}[15x^2-2\sqrt{105}x+14]$, when x is put equal to $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{15}}$.

4. If the ratio $2a+b:2a-b$ is the duplicate of the ratio $c+2a:c-2a$, then the ratio $2c-b:b$ is the duplicate of the ratio $c:2a$.

5. Solve $x^3-y^3=19$, $\dfrac{x}{y}-\dfrac{y}{x}=\dfrac{5}{6}$.

6. Define an Arithmetical Progression; and given the first and last of $(2n+1)$ numbers in A. P. find the middle number.

Two terms of an A. P. are 48 and 106, and there are 8 terms between them; find the term which is arithmetically the least in the series of which these eight terms form a portion.

7. The first term of a series of numbers is $x+\dfrac{1}{2x}$, and the n^{th} is $nx+\dfrac{1}{2^n x}$. Find the sum of n terms.

If the third term is equal to the fourth, find x.

---

8. Find the H. C. F. of

$$x^5-2x^4-2x^3+8x^2-7x+2 \text{ and } x^4-4x+3.$$

If A and B are two quantities and C is their H. C. F., show that their L. C. M. is $\dfrac{AB}{C}$.

9. Solve:

(i) $x^2-2x\sqrt{3}-13=0$.

(ii) $x+y=5$, $x^2-2y^2=1$.

10. Prove that

$$\frac{1}{x-1}+\frac{1}{x-3}+\frac{1}{x-5}+\frac{1}{x-7}=0,$$

when $x=4+\sqrt{5}$.

For what other values will the statement be true?

11. (i) If $a:b::c:d$, prove that $a+b:a-b::c+d:c-d$.

(ii) If $\frac{x}{a}=\frac{y}{b}$, prove that each fraction $=\frac{lx+my}{la+mb}$.

(iii) If $\frac{x}{a}>\frac{y}{b}$, prove that $\frac{lx+my}{la+mb}$ is intermediate in value between $\frac{x}{a}$ and $\frac{y}{b}$.

12. Find the sum of n terms of an Arithmetic Progression of which the first term and the common difference are given.

If the first term is $3N+1$ and the common difference is -6, find the number of the positive terms, and their sum (i) when N is even, (ii) when N is odd.

13. Sum the following series:

(i) $108+101+99+\ldots$ to 108 terms;

(ii) $3-1+\frac{1}{3}+\ldots$ to infinity.

14. The incomes of A and B are in the ratio of 3 to 2 and their expenditures are in the ratio of 5 to 3. Each saves £1000 a year. Find their incomes.

---

15. Show that if $a+b+c+d=0$, then

$$a^3+b^3+c^3+d^3=3abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

16. Solve:

(i) $\frac{a}{x+b}+\frac{b}{x+a}+2=0$;

(ii) $x(x-y)=10$,
$y(x+y)=24$.

17. Prove that $a^m\div a^n=a^{m-n}$, where m, n are positive integers, and m is greater than n.

Simplify $\left(\frac{x^4}{y^{\frac{1}{3}}}\right)^{-\frac{1}{2}}\times\left(\frac{x^{-\frac{3}{2}}}{x^{-\frac{1}{2}}}\right)^4\div(x^{-1}y^2z^{-3})^2$.

18. If α and β are the roots of the equation

$$ax^2+bx+c=0,$$

find the equation whose roots are $\alpha+\alpha^{-1}$, $\beta+\beta^{-1}$.

19. If $a:b::c:d$, show that

$$\sqrt{ac}+\sqrt{bd}:\sqrt{ac}-\sqrt{bd}=(a+b)\,d:(c-d)\,b.$$

20. If P, Q, R are the p^{th}, q^{th}, r^{th} terms of an A. P., prove that

$$P(q-r)+Q(r-p)+R(p-q)=0.$$

21. A hare runs ten times as fast as a tortoise which it is pursuing. The hare at A is 100 yards behind the tortoise at B; when the hare is at B the tortoise has advanced to C, and when the hare is at C the tortoise has advanced to D, and so on an endless number of times. Find, by adding AB, BC, CD, &c., how far the hare will have gone when it overtakes the tortoise.

---

22. If a rational integral polynomial in x vanishes when $x=a$, show that it contains $x-a$ as a factor.

Show that

$$bc(b-c)+ca(c-a)+ab(a-b)+(b-c)(c-a)(a-b)=0.$$

23. Find the first three terms of a series of ascending powers of x which, when multiplied by $1-\frac{1}{2}x^2+\frac{1}{4}x^4$, is equal to unity.

24. Solve:

(i) $\dfrac{1}{x+a}-\dfrac{1}{x+b}=\dfrac{1}{x+a+1}-\dfrac{1}{x+b+1}$;

(ii) $6x^2-11xy+4y^2=0, \quad x+2y=5.$

Verify graphically.

25. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each fraction $=\sqrt{\dfrac{ce+ea+ac}{df+fb+bd}}$ and also $=\dfrac{adf+cfb+ebd}{3bdf}$.

26. (i) A series in Arithmetic Progression consists of n terms; its first term is $n-1$ and its last term is $1-n$. Find the last term but one, and the sum of the series.

(ii) Sum to n terms the series

$$2-4+8-16+\ldots.$$

27. If α, β, γ are any three roots of the equation

$$ax^4+bx^3+cx-a=0,$$

prove that $\beta\gamma+\gamma\alpha+\alpha\beta=\dfrac{1}{\beta\gamma}+\dfrac{1}{\gamma\alpha}+\dfrac{1}{\alpha\beta}$.

28. What digit must the symbol x denote, in order that the product of the two numbers $31x$ and $3x1$ may equal $11x\,60x$?

---

29. Explain what is meant by (i) an identity, (ii) by an equation.

Why is the sign of a quantity changed when it is transferred from one side of an equation to the other?

A quadratic function of x vanishes when $x=2$, is equal to -1 when $x=1$, and is equal to 4 when $x=-2$. What is its value when $x=-1$?

30. Simplify $\dfrac{\{(x^m)^{\frac{1}{r}}(x^p)^{\frac{1}{n}}\}^{nr}}{\{\sqrt{a^n}\times\sqrt[m]{a^r}\}^{mp}}\times\left\{\left(\dfrac{a}{x}\right)^p\right\}^{r-1}$.

31. Simplify:

(i) $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}$;

(ii) $\Sigma\left\{\dfrac{a}{(a-b)(a-c)(a-x)}\right\}$.

32. Solve:

(i) $(a^2+ab-2b^2)x^2-(2a^2+ab+3b^2)x+a^2-b^2=0$;

(ii) $x^2(1+2y)=y+2x=y^2(2+x)$.

33. Explain what is meant by the expression 'sum to infinity' of a geometrical progression. Under what circumstances can such a sum be obtained?

Sum to n terms and to infinity the series

$$8-4\sqrt{2}+4-2\sqrt{2}+\ldots;$$

and to n terms the series

$$1+3+7+15+31+\ldots.$$

34. The value of diamonds $\propto$ the square of their weights and the square of the value of rubies $\propto$ the cube of their weights; a diamond of a carats is worth m times a ruby of b carats, and both together are worth £c; find the value of a diamond and ruby each weighing x carats.

35. Two particles are projected, in opposite directions (towards one another), from the ends of a straight tube, 268 inches long. One passes over 20 inches in the first second, 18 inches in the second second, 16 inches in the third second, and so on; the other passes over 24 inches in the first second, 23 inches in the second second, 22 inches in the third second, and so on. In what time will they meet?

36. Prove the identity

$$a^4+b^4+(a+b)^4 \equiv 2a^2b^2+2(a^2+b^2)(a+b)^2.$$

If $x+y+z=0$, prove that

$$x^4+y^4+z^4 \equiv 2y^2z^2+2z^2x^2+2x^2y^2.$$

37. Divide x^6+y^{-6} by $x^3+2^{\frac{1}{2}}x^{\frac{3}{2}}y^{-\frac{3}{2}}+y^{-3}$.

38. If $\dfrac{y+z-x}{a}=\dfrac{z+x-y}{b}=\dfrac{x+y-z}{c}$, prove that

$$\frac{a+b+c}{x+y+z}=\frac{ay+bz+cx}{x^2+y^2+z^2}.$$

39. Find the H. C. F. of

$15x^4-10x^3y+4x^2y^2-6xy^3-3y^4$ and $6x^3-14x^2y+3xy^2+5y^3$.

State carefully the reasons for each step in the process that you employ.

40. State the condition necessary that one quantity may be the simple fraction $\dfrac{m}{n}$ of another quantity, and prove that $\dfrac{m}{n}=\dfrac{km}{kn}$ where k, m, n are integers. §134.

41. Define a series of terms in geometrical progression; and find a formula for the sum of n terms when the first term is a and the second b.

The geometric mean between two numbers is 12 and the arithmetic mean is $25\frac{1}{2}$. Find the two numbers and their harmonic mean.

42. Sum to n terms:

(i) $a+(2a+b)x+(3a+2b)x^2+(4a+3b)x^3+\ldots$;

(ii) $\dfrac{3}{1^2.2^2}+\dfrac{5}{2^2.3^2}+\dfrac{7}{3^2.4^2}+\dfrac{9}{4^2.5^2}+\ldots$.

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43. Prove that $\Sigma a^2(b-c)=\Pi(b-c)$, there being only three letters a, b, c involved.

44. Explain the words logarithm, mantissa, characteristic.

What great advantage is there in taking 10 as the base of logarithms?

Prove that $\log(x^m y^n)=m\log x+n\log y$.

If $a^2+b^2=c^2$, prove that $\dfrac{1}{\log_{c+b}a}+\dfrac{1}{\log_{c-b}a}=2$.

45. (i) Simplify $\dfrac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-(a^{\frac{1}{2}}+b^{\frac{1}{2}})^4$;

(ii) Express $\sqrt{158+60\sqrt{6}}$ as the sum of two surds.

46. Solve $x+y+z=1$,
$x^2+y^2+z^2=9$,
$x^3+y^3+z^3=1$.

47. A man receives a pension, starting with £100 the first year, but each year he receives 90 per cent. of what he received the previous year. Find the total amount he receives in the first 6 years, in pounds and decimals of a pound; find also the greatest amount he could possibly receive even if he were to live for ever.

48. Prove by Mathematical Induction:

(i) $\Sigma(n^2)=\dfrac{n(n+1)(2n+1)}{6}$;

(ii) $\Sigma(n^3)=\left(\dfrac{n(n+1)}{1.2}\right)^2$.

Find the sum of n terms of the series whose n^{th} term is

$$n(n+1)(n+2).$$

49. Find the number of balls in a complete rectangular pile, the length and breadth of the base containing 52 and 34 balls respectively.

50. Graph the function $\frac{1}{4}(3x^2-4x-7)$ from $x=-3$ to $x=2$.
Find the minimum value of the function. Find the slope of the curve when $x=2$.

CHAPTER XXVII

PERMUTATIONS AND COMBINATIONS

273. By actual multiplication it can be shown that

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc,$$

$$(x+a)(x+b)(x+c)(x+d)=x^4+(a+b+c+d)x^3$$
$$+(ab+ac+ad+bc+bd+cd)x^2+(bcd+acd+abd+abc)x+abcd.$$

In this last the coefficient of x^3 is the sum of all possible ways of taking one letter out of the four letters a, b, c, d; the coefficient of x^2 is the sum of all possible ways of taking the product of two letters out of the four letters a, b, c, d; similarly the coefficient of x is obtained by taking the letters three at a time; and the constant term by taking them four at a time.

ab is called a combination of the letters taken two at a time.

bcd is called a combination of the letters taken three at a time.

&c., &c.

If now $a = b = c = d$, the number of terms in any coefficient is the same as the number of combinations that can be made out of four letters taken 1, 2, 3, or 4 at a time so that

$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4.$$

Similarly, results hold for n factors, but before considering the general formula for $(x+a)^n$, we must discuss methods for finding the number of combinations of n things taken r together.

274. Example. *There are three niches in the front of a building; the architect has four statues which he can use to fill the niches. In how many ways can the niches be filled?*

The first niche can be filled in 4 ways.

When the first is filled, 3 statues are left.

Therefore corresponding to each way of filling the first niche there are 3 ways of filling the second niche. Hence the number of ways of filling the first two niches is 4×3.

There are now only 2 statues left; therefore the remaining niche can be filled in two ways.

Hence the number of ways of filling all three niches $= 4 \times 3 \times 2 = 24$.

The student should verify this by actually making all possible arrangements of three letters out of the four letters a, b, c, d. In this example we have had to consider not only the fact that 3 statues are to be taken out of 4, but also that there are different arrangements of the same 3 statues.

275. Each arrangement that can be made by choosing r things out of n things is called a **Permutation.**

The symbol ${}_nP_r$ is used to denote the number of Permutations of n things taken r at a time.

Each group of r things that can be chosen out of n things is called a Combination, the symbol ${}_nC_r$ is used to denote the number of Combinations of n things taken r at a time.

In dealing with Combinations we consider only the group of r things, but with Permutations we consider arrangement as well.

Examples XXVII a.

By actually writing down the Permutations and Combinations of a, b, c, d, &c., prove that:

1. ${}_5C_2 = 10$, ${}_5P_2 = 20$.

2. ${}_4C_3 = 4$, ${}_4C_1 = 4$, ${}_4P_1 = 4$, ${}_4P_3 = 24$.

3. ${}_6C_2 = 15$, ${}_6P_2 = 30$.

4. ${}_5C_3 = 10$, ${}_5P_3 = 60$.

5. How many different numbers of 4 digits can be made (i) using no digit more than once, (ii) using any digit as often as we please?

6. In how many ways can 5 boys be arranged on a form?

7. In how many ways can the letters of the word *thimble* be arranged? How many ways are there if the vowels are kept separate?

8. Show that ${}_6P_4 = 6 \times {}_5P_3$, and by a general argument show that ${}_{n+1}P_{r+1} = (n+1){}_nP_r$ (without actually finding ${}_{n+1}P_{r+1}$ or ${}_nP_r$).

9. Show that ${}_7C_6 = {}_7C_1$, and by a general argument show that ${}_nC_r = {}_nC_{n-r}$ (without actually finding ${}_nC_r$ or ${}_nC_{n-r}$). Show also that ${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$.

10. Find the number of ways in which a half-crown, a shilling, a sixpence, and a penny can be distributed among six boys (i) when no boy is allowed to receive more than one coin, and (ii) when there is no restriction as to the number of coins any boy may receive?

11. Show that the number of ways of arranging n things taken all together is $\lfloor\underline{n}$ (see § 247, Example III).

12. Find the number of arrangements that can be made of the 5 letters a, a, b, c, d taken all together.

13. Find the number of permutations that can be made of 7 letters of which 3 are alike, all the letters being used for each permutation.

14. Show that ${}_nP_r$ equals the product of n consecutive integers of which n is the greatest.

15. Show that the number of ways of arranging n letters of which r are alike is $\frac{\lfloor n}{\lfloor r}$.

16. In how many different ways can 7 people arrange themselves in a circle?

17. If x is the number of combinations of 3 things that can be formed from n things, show that the number of permutations equals $x\lfloor 3$.

276. To show that

$$ {}_nP_r = n(n-1)(n-2)\ldots(n-r+1). \quad (r \text{ factors}). $$

The first place may be filled in n ways.

When it is filled the second place may be filled in $n-1$ ways.

Hence the first two places may be filled in $n(n-1)$ ways.

The third place may now be filled in $n-2$ ways.

Hence the first three places may be filled in $n(n-1)(n-2)$ ways.

And similarly, by induction, r places may be filled in $n(n-1)\ldots(n-r+1)$ ways.

Alternative method.

Take one thing away; the remaining $n-1$ things may be arranged $r-1$ together in ${}_{n-1}P_{r-1}$ ways.

Replace the n^{th} thing before each of these permutations;

then there are ${}_{n-1}P_{r-1}$ permutations in which a particular thing stands first.

But there are n things in all;

$\therefore$ there are $n\,{}_{n-1}P_{r-1}$ permutations in all, each consisting of r letters, the choice being made from n letters;

$$ \text{i.e.} \quad {}_nP_r = n\,{}_{n-1}P_{r-1}. $$

Similarly,
$$_{n-1}P_{r-1} = (n-1)\,{}_{n-2}P_{r-2},$$
$$_{n-2}P_{r-2} = (n-2)\,{}_{n-3}P_{r-3},$$
$$_{n-r+2}P_2 = (n-r+2)\,{}_{n-r+1}P_1,$$
$$_{n-r+1}P_1 = (n-r+1),$$

since $n-r+1$ things can be arranged in $n-r+1$ ways when only 1 is taken at a time.

On multiplying, all the P's cancel except ${}_nP_r$;

$$\therefore\ {}_nP_r = n\,(n-1)\,(n-2)\ldots(n-r+1),$$

or ${}_nP_r$ = the product of r consecutive integers of which n is the greatest.

If all the n things are used for each permutation, we have the case where $r = n$. Hence the number of permutations of n things when all are taken $= n\,(n-1)\ldots n$ factors.

$$= \lfloor\underline{n}.$$

277. Example. *In how many ways can* 5 *letters be arranged, of which* 3 *are a's, and the others are different?*

Let x be the required number of ways.

Consider any one arrangement, say $a\ b\ a\ a\ c$.

Suppose now the a's were distinguished as $a_1\ a_2\ a_3$; then from this arrangement we could make 6 (i.e. $\lfloor\underline{3}$) other arrangements by permuting the a's, viz. $a_1\ b\ a_2\ a_3\ c$, $a_1\ b\ a_3\ a_2\ c$, &c.

And this could be done for each of the x arrangements.

Therefore if the a's were all different there would be $x \times \lfloor\underline{3}$ arrangements.

But as the 5 letters are now all different, the number of permutations $= \lfloor\underline{5}$.

Hence
$$x \times \lfloor\underline{3} = \lfloor\underline{5},$$
$$x = \frac{\lfloor\underline{5}}{\lfloor\underline{3}} = 5\,.\,4 = 20.$$

The general question can be solved the same way.

278. To find the number of ways in which n things can be arranged if p are alike of one kind, q of another kind, r of another....

Let x denote the number of arrangements.

Suppose the p like things be replaced by p unlike things.

These can be permuted among themselves in $\lfloor p$ ways; so that any one of the previous arrangements can be replaced by $\lfloor p$ different arrangements. There would now be $x \times \lfloor p$ arrangements.

Similarly, if the q like things were replaced by q unlike things, we should multiply by $\lfloor q$; and so for any other group of like things.

Hence when all are unlike the number of permutations would be $x \times \lfloor p \times \lfloor q \times \lfloor r \ldots$.

But when all are unlike the number of permutations is $\lfloor n$;

$$\therefore \quad x \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n;$$

i.e. required number of arrangements $= \dfrac{\lfloor n}{\lfloor p \times \lfloor q \times \lfloor r}$.

279. To find the number of combinations of n things taken r at a time.

Method I. Let x be the required number.

Any particular combination may be permuted in $\lfloor r$ ways, so that if all the combinations are permuted there will be $x \times \lfloor r$ permutations.

But if all the combinations are permuted, we have all the possible permutations of n things taken r at a time;

$$\therefore \quad x \times \lfloor r = {}_nP_r = n(n-1) \ldots (n-r+1).$$

But x is ${}_nC_r$.

Hence $$ {}_nC_r = \frac{n(n-1) \ldots (n-r+1)}{\lfloor r}.$$

Method II. If we first take a group of $r-1$ things, there will be $n-r+1$ things left. A group of r things is obtained by adding any one of those left to the group of $r-1$.

The possible number of groups of $r-1$ things is ${}_nC_{r-1}$;

$(n-r+1)$ groups of r can be obtained from each of these.

This would give $(n-r+1)\,{}_nC_{r-1}$ groups of r things.

But in this number each group is counted r times, once for each of its r constituents.

Hence $$ {}_nC_r = (n-r+1)\,{}_nC_{r-1} \div r$$

i.e. $$ {}_nC_r = \frac{(n-r+1)}{r}\,{}_nC_{r-1}.$$

Similarly, by continually changing r into $r-1$,

$$ {}_nC_{r-1} = \frac{n-r+2}{r-1}\,{}_nC_{r-2}, $$

$$ {}_nC_{r-2} = \frac{n-r+3}{r-2}\,{}_nC_{r-3}, $$

$$ {}_nC_2 = \frac{n-1}{2}\,{}_nC_1; $$

but $$ {}_nC_1 = n. $$

Multiplying $$ {}_nC_r = \frac{n(n-1)\dots(n-r+1)}{1.2\dots r}. $$ *

280. **Alternative forms for ${}_nP_r$ and ${}_nC_r$.**

$$ {}_nP_r = n(n-1)\dots(n-r+1) \times \frac{(n-r)(n-r-1)\dots 3.2.1}{(n-r)(n-r-1)\dots 3.2.1} $$

$$ = \frac{\lfloor n}{\lfloor n-r}, $$

$$ {}_nC_r = \frac{{}_nP_r}{\lfloor r} \qquad (\S\ 279) $$

$$ = \frac{\lfloor n}{\lfloor n-r \,\lfloor r}. $$

It is easily deduced that

(i) $\lfloor 0 = 1$ by putting $r = n$ in the two formulae for ${}_nP_r$.

(ii) ${}_nC_r = {}_nC_{n-r}$.

(iii) ${}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1}$.

281. To find number of arrangements of n things, of which p are alike of one kind, q of another, r of another, &c.; all the n things being used for each arrangement.

* The argument is rendered clearer by an example. Consider the number of combinations of the five letters *abcde* taken 4 together.

A 3-combination is *abc*, from which are formed $\frac{abcd}{abce}$.

Similarly, from each 3-combination two 4-combinations can be formed, this would give ${}_5C_4 = 2 \times {}_5C_3$.

But the 4-combination *abcd* can be formed from the 3-combination *abc* by adding *d*, from *abd* by adding *c*, from *acd* by adding *b*, and from *bcd* by adding *a*; so that it has been counted 4 times; hence we must divide by 4.

$$ \therefore\ {}_5C_4 = \frac{2}{4} \times {}_5C_3, \text{ i.e. } = \frac{5-4+1}{4}\,{}_5C_3. $$

Alternative proof.

Consider that the n things are to be put in n places.

A group of p places can be chosen in ${}_nC_p$ ways. Fill this group with the p like things.

There remain $n-p$ places; a group of q can be chosen in ${}_{n-p}C_q$ ways. Fill this group with the q like things.

Similarly, ${}_{n-p-q}C_r$ groups of r places can be filled with the r like things.

There remain $n-p-q-r$ places, the remaining unlike things can be arranged in these in $\lfloor n-p-q-r$ ways.

Hence required number of arrangements

$$= {}_nC_p \times {}_{n-p}C_q \times {}_{n-p-q}C_r \times \lfloor n-p-q-r$$

$$= \frac{\lfloor n}{\lfloor p \lfloor n-p} \cdot \frac{\lfloor n-p}{\lfloor q \lfloor n-p-q} \cdot \frac{\lfloor n-p-q}{\lfloor r \lfloor n-p-q-r} \lfloor n-p-q-r = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

282. Example I. *In how many ways can 7 people arrange themselves at a round table?*

Let any one of them sit down.

The place on his right can be filled in 6 ways.

The place on the second person's right can be filled in 5 ways.

The place on the third person's right can be filled in 4 ways;

and so on.

Number of arrangements $= 6.5.4.3.2.1 = 720$.

If the position of the people with respect to the room is considered, the number of ways of sitting down is $\lfloor 7$, as in the case when the 7 people sit in a straight line.

If 7 different coloured beads are arranged on a circular wire there are only $\frac{\lfloor 6}{2}$ different arrangements. Why?

Example II. *In how many ways may seven persons distribute themselves among the ten seats of a compartment of a railway carriage, if three of them refuse to ride with their backs to the engine?*

The three have 5 places to choose from, and may take their seats in ${}_5P_3$ ways.

There are now 7 places left for the remaining four, who may, therefore, choose their seats in ${}_7P_4$ ways.

Hence the seven may seat themselves in ${}_5P_3 \times {}_7P_4$ ways.

Number of ways $= 5.4.3 \times 7.6.5.4$

$= 60 \times 7 \times 120$

$= 50400.$

Example III. *Find the number of different words that can be formed by using all the letters in the word repetition, there being always a consonant both in the first and last place.*

There are three possible sets of cases to consider: (i) *t* at beginning and end, (ii) *t* at beginning or end but not both, (iii) *t* neither at beginning nor end.

(i) When the two *t*'s are at beginning and end, there remain 8 places to be filled by 8 letters of which 2 are *e*'s, 2 are *i*'s;

$\therefore$ remaining places can be filled in $\frac{\lfloor 8}{\lfloor 2 \lfloor 2} = 10080$ ways.

(ii) When *t* is at beginning, the end may be filled in 3 ways.

Hence there are $2 \times 3 = 6$ ways of having *t* either at beginning or end.

And as in Case I, the remaining 8 places can be filled in 10080 ways.

Hence there are altogether 6×10080 words with *t* at one end.

(iii) There are ${}_3P_2$ ways ($= 6$ ways) of filling the first and last places without using *t*.

There remain 8 places to be filled by 8 letters of which 2 are *e*'s, 2 are *i*'s, 2 are *t*'s;

$\therefore$ remaining places can be filled in $\frac{\lfloor 8}{\lfloor 2 \lfloor 2 \lfloor 2} = 5040$ ways.

Hence there are altogether 6×5040 words with *t* neither at beginning nor end.

Therefore total number of words

$$= 10080 + 6 \times 10080 + 6 \times 5040$$
$$= 10 \times 10080$$
$$= 100800.$$

Example IV. *A person has* 20 *acquaintances,* 12 *of whom are relatives. In how many ways may he invite* 15 *guests from among them so that exactly* 8 *of these are relatives?*

He can choose 8 relatives in ${}_{12}C_8$ ways.

He can choose the remaining 7 guests in ${}_8C_7 = {}_8C_1$ ways.

But each set of 8 relatives may be asked with each set of 7 acquaintances.

$\therefore$ Number of ways required $= {}_{12}C_8 \times {}_8C_1$

$$= \frac{12.11.10.9}{1.2.3.4} \times 8 \qquad (\text{since } {}_{12}C_8 = {}_{12}C_4)$$

$$= 3960.$$

Example V. *Find the value of r in order that ${}_8C_r$ may have its greatest value.*

$${}_8C_r = \frac{8.7\ldots(8-r+2)(8-r+1)}{\lfloor r},$$

$${}_8C_{r-1} = \frac{8.7\ldots(8-r+2)}{\lfloor r-1}.$$

Hence $${}_8C_r = {}_8C_{r-1} \times \frac{9-r}{r};$$

$$\therefore\ {}_8C_r > {}_8C_{r-1},$$

so long as $$\frac{9-r}{r} > 1;$$

i.e. $$9-r > r;$$

i.e. $$9 > 2r;$$

$\therefore$ ${}_8C_r$ continues to increase as r increases until $r = 4$.

$\therefore$ ${}_8C_r$ is greatest when $r = 4$.

Example VI. *A person throws with two dice, is he more likely to throw a total* 8 *or* 7 *?*

For a total 8 he must throw two 4's which can be done in 1 way;
or 5 and 3 which can be done in 2 ways;
or 6 and 2 which can be done in 2 ways.

$\therefore$ there are 5 ways of throwing a total 8.

For a total 7 he must throw 4 and 3 which can be done in 2 ways;
or 5 and 2 which can be done in 2 ways;
or 6 and 1 which can be done in 2 ways.

$\therefore$ there are 6 ways of throwing a total 7.

Hence he is more likely to throw 7.

Example VII. *Five boys from a certain school are candidates for the Army. In how many ways can a selection be made?*

There are 2 ways of dealing with each boy since he may be chosen or rejected. Either way of dealing with 1st boy may be taken with either way of dealing with another boy.

Hence there are 2×2 ways of dealing with 2 boys.

Similarly, there are 2^3 ways of dealing with 3 boys;

and there are $2^5 = 32$ ways of dealing with 5 boys.

This might have been answered as follows:—

5 boys might be chosen in 1 way;
4 boys might be chosen in ${}_5C_4 = 5$ ways;
3 boys might be chosen in ${}_5C_3 = 10$ ways;
2 boys might be chosen in ${}_5C_2 = 10$ ways;
1 boy might be chosen in ${}_5C_1 = 5$ ways;
0 boys might be chosen in 1 way.

$\therefore$ Total number of ways = 32.

Examples XXVII.

1. If ${}_nP_3 : {}_nP_4 :: 1 : 6$, find n.

2. If ${}_nP_3 = 12 \times {}_{\frac{n}{2}}P_3$, find n.

3. How many different arrangements may be made by using all the letters of the word *Mississippi*?

4. How many different signals can be formed by means of 12 different flags which can be hoisted 4 at a time in a vertical line?

5. If ${}_nP_3 = 6\,{}_nC_4$, find n.

6. In how many ways can 6 men be chosen for promotion out of 15 candidates?

7. How many different sums can be formed with 5 of the following coins: a sovereign, a half-sovereign, a crown, a half-crown, a shilling, a sixpence, a penny, a farthing?

What is the value of all these sums added together?

8. How many hands of 3 cards is it possible to deal to a person with an ordinary pack of playing cards?

How many of these are all hearts?

9. Explain the difference between permutations and combinations.

If the number of combinations of a certain number of things taken 5 together is to the number of their permutations taken 3 together as 5 : 1, find the number of things.

10. Find the greatest value of ${}_nC_r$ when $n = 9$ and when $n = 10$.

11. A schoolmaster has 11 boarders. He purposes giving extra tuition to r of them every evening, and to continue doing so as long as he can have a different group each time. Find the greatest time these lessons can continue. How many times would each boy attend?

12. A man has 5 coats, 6 waistcoats, and 4 pairs of trousers. On how many days can he put on a different set of clothes?

13. Find the greatest value of ${}_nC_r x^r$ when $n = 7$, $x = 3$.

14. A party of 12, consisting of 6 husbands and their wives, meet to play bridge once a week for a period of six weeks. They wish to arrange that no person shall have the same partner twice or play against the same person twice. Is this possible? Give reasons.

15. Prove in any way that

$$ {}_{n+2}C_r = {}_nC_r + 2{}_nC_{r-1} + {}_nC_{r-2}. $$

16. How many different numbers can be made out of the figures 1220005555?

17. In how many ways can 5 sixpences be distributed among 4 boys?

18. In how many ways can a foot-rule be divided into three pieces so that each piece contains an exact number of inches?

19. Find the number of triangles into which a polygon of m sides can be divided by joining the angular points.

20. Find the number of permutations of the letters of the word December taken all together, and also the number of combinations four at a time.

21. Twenty-six people wish to travel by an omnibus which holds 12 inside and 14 outside. In how many ways can (i) the inside, (ii) the outside passengers be selected?

22. What is the answer to the previous question if 8 refuse to ride outside and 6 will not go inside?

23. Find the number of permutations of the letters forming the word *treasurers* taken together.

24. In how many ways can a picket of 3 men and an officer be chosen out of a company of 80 men and 3 officers?

25. A committee of eight is to be selected from 14 men, of whom six are British and eight are Canadians. In how many ways can the committee be selected so that the British may not be outnumbered?

26. Seven prizes are to be distributed amongst five boys, each boy being eligible for all the prizes. In how many ways can they be given away?

27. In how many ways can 12 articles be arranged in three parcels so that the first contains 3 articles, the second 4, and the third 5?

28. A closed chain is formed of twelve links of different metals, how many distinct arrangements are there?

29. A straight line is divided into n segments. Each part is to be coloured black, blue, or red. How many ways are there of shading the line (1) if no restriction is imposed, and (2) if no two consecutive segments are to be of the same colour?

30. Twenty-two men arrange to play a cricket match. If two of the men are brothers, show that the number of ways in which the teams can be made up so that the brothers do not play on the same side is $\dfrac{2\,\lfloor\underline{19}}{\lfloor\underline{10}\,\lfloor\underline{9}}$.

31. Show that if n and r are positive integers, and n is greater than r, then $\dfrac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}$ is an integer.

32. Does $\lfloor\underline{0} = 0$ or 1 or neither? Give reasons.

33. A railway carriage can seat five each side. Find in how many different ways a party of four ladies and six gentlemen can seat themselves so that the ladies always have the corner seats.

34. Find the total number of different combinations (any number at a time) of the five letters x, x, y, y, z.

35. Show that the total number of different combinations (any number at a time) of n letters of which p are alike of one kind, q of another, and the rest are unlike, is $(p+1)(q+1)\,2^{n-p-q}-1$.

36. How many different numbers greater than a million can be made with the digits 0, 1, 2, 2, 2, 3, 3?

37. Find the number of permutations of the letters of the word *zoology* taken all together; and find the number of ways in which a selection of four letters can be made from the letters of this word.

38. A person wishes to make up as many different parties as he can out of 20 friends, each party consisting of the same number, how many should he invite at a time?

39. In how many ways can 12 persons be divided into three sets of four each, one set to play lawn-tennis, one to play croquet, and one to play bowls?

40. How many numbers are there consisting of 5 different digits arranged in ascending order of magnitude?

41. A certain football club contains 20 members, of whom 11 can only play forward, and 9 can only play back. In how many ways can a team of 8 forwards and 7 backs be selected?

42. Find the number of combinations of n things taken r together, without assuming the number of permutations. Find for what value of r ${}_{10}C_r$ is greatest.

43. A dinner party consisting of ten persons sit round a table, a lady and gentleman alternately: the host and hostess each occupying one end of the table, and their guests being arranged four on each side. Find the number of ways in which the table may be arranged.

44. Prove from the formula that

$${}_nC_r = {}_nC_{n-r} \text{ and } {}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1}.$$

45. Prove (i) $1 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_r + \ldots + {}_nC_n = 2^n$;

(ii) $1 + {}_nC_2 + {}_nC_4 + \ldots = 2^{n-1}$;

(iii) ${}_nC_1 + {}_nC_3 + {}_nC_5 + \ldots = 2^{n-1}$.

46. In a bag are 7 red balls and 4 white balls; in another bag are 6 red balls and 5 white balls. A draws two balls from the first bag, and B two balls from the second. Show that B is more likely to draw one red and one white than A is. If now, one black ball is added to the first bag and two black balls to the second, which is the more likely to draw one red and one white?

47. How many terms are there in the product:—

(i) $(x_1 + a_1)(x_2 + a_2) \ldots n$ factors?

(ii) $(x + a_1)(x + a_2) \ldots n$ factors?

(iii) $(x + a)(x + a) \ldots n$ factors?

48. How many terms each containing 2 x's and 3 a's can be formed from the ten letters, $x_1, x_2, x_3, x_4, x_5, a_1, a_2, a_3, a_4, a_5$? How many can be formed if no subscript can occur twice in the same product? Hence find the coefficient of $a^3 x^2$ in $(a+x)^5$.

CHAPTER XXVIII

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

283. To find the expanded form of $(x+a)^n$, without actual multiplication, when n is a positive integer.

(i) Since n is a positive integer,

$$(x+a)^n = (x+a)(x+a)+\ldots n \text{ factors.}$$

(ii) Each letter in each bracket is multiplied by each letter in every other bracket.

(iii) Therefore every term in the product is of n dimensions, and is of the form $a^r x^{n-r}$.

(iv) x may be taken from each bracket giving x^n; and this can be done in only 1 way.

x may be taken from $n-1$ brackets and a from the remaining bracket; since there are n a's to choose from, this can be done in ${}_nC_1$ ways.

Hence there are ${}_nC_1$ terms, each equal to ax^{n-1}.

x may be taken from $n-2$ brackets, and a from the remaining 2 brackets.

2 a's can be chosen in ${}_nC_2$ ways.

Hence there are ${}_nC_2$ terms, each equal to $a^2 x^{n-2}$, and generally there are ${}_nC_r$ terms each equal to $a^r x^{n-r}$.

The last term is obtained by taking a from each bracket.

This can be done in only 1 way.

Hence the last term is a^n.

(v) We have, therefore,

$$(x+a)^n = x^n + {}_nC_1 ax^{n-1} + {}_nC_2 a^2 x^{n-2} + \ldots + {}_nC_r a^r x^{n-r} + \ldots + a^n.$$

If we substitute the values for ${}_nC_1$, ${}_nC_2$, &c., we have

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}a^r x^{n-r} + \ldots + a^n.$$

This is known as the **Binomial Theorem.**

284. Proof of Binomial Theorem by Induction.

It is required to prove that

$$(x+a)^n = x^n + {}_nC_1 ax^{n-1} + \ldots + {}_nC_r a^r x^{n-r} + \ldots + a^n.$$

We know from actual multiplication that

$$(x+a)^2 = x^2 + 2ax + a^2 = x^2 + {}_2C_1 ax + a^2;$$

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 = x^3 + {}_3C_1 ax^2 + {}_3C_2 a^2x + a^3.$$

The theorem is, therefore, true when $n = 2$ or 3.

Assume the theorem to be true for index n.

Then $(x+a)^{n+1}$

$$= (x^n + {}_nC_1 ax^{n-1} + \ldots + {}_nC_{r-1}a^{r-1}x^{n-r+1} + {}_nC_r a^r x^{n-r} + \ldots + a^n) \times (x+a)$$

$$= x^{n+1} + {}_nC_1 ax^n + \ldots + {}_nC_{r-1}a^{r-1}x^{n-r+2} + {}_nC_r a^r x^{n-r+1} + \ldots + a^n x + ax^n + \ldots + {}_nC_{r-1}a^r x^{n-r+1} + \ldots + a^{n+1}.$$

$$= x^{n+1} + (n+1)ax^n + \ldots + ({}_nC_r + {}_nC_{r-1})a^r x^{n+1-r} + \ldots + a^{n+1}$$

$$= x^{n+1} + {}_{n+1}C_1 ax^n + \ldots + {}_{n+1}C_r a^r x^{n+1-r} + \ldots + a^{n+1},$$

since $${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r; \quad (\S\ 280\ (iii))$$

$\therefore$ if theorem is true for index n, it is also true for index $n+1$.

But it has been shown to be true for index 2 and index 3.

Therefore it is true when $n = 4$, and so on.

Hence the theorem is proved when n is any positive integer.

285. The following details should be noted; the proofs are left as easy exercises for the student.

(i) The first and last terms can both be derived from the general term ${}_nC_r a^r x^{n-r}$ by giving the proper values to r.

(ii) Terms equidistant from the first and last terms have equal coefficients.

(iii) If a is negative, the terms are alternately positive and negative.

(iv) There are $(n+1)$ terms.

(v) The sum of all the coefficients is 2^n.

(vi) The sum of the odd coefficients (i.e. the coefficients of the odd terms) is equal to the sum of the even coefficients.

286. ${}_nC_r a^r x^{n-r}$ is called the **General Term.**

It should be noticed that the general term is the $\overline{r+1}^{\text{th}}$ term and not the r^{th} term.

It is useful to recollect the expanded form, viz.

$$(r+1)^{\text{th}} \text{ term} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r} \cdot a^r x^{n-r}.$$

The general term of $(x-a)^n$ is ${}_nC_r(-a)^r x^{n-r}$, which equals $(-1)^r {}_nC_r a^r x^{n-r}$.

287. Example I. *Expand* $(3-2x)^6$.

$$(3-2x)^6 = 3^6 - {}_6C_1 . 3^5 . (2x) + {}_6C_2 . 3^4 . (2x)^2 - {}_6C_3 . 3^3 (2x)^3 + {}_6C_4 3^2 . (2x)^4 - {}_6C_5 . 3(2x)^5 + (2x)^6$$

$$= 729 - 6 . 243 . 2x + \frac{6.5}{1.2} \cdot 81 . 4x^2 - \frac{6.5.4}{1.2.3} \cdot 27 . 8x^3 + \frac{6.5}{1.2} \cdot 9 . 16x^4 - 6 . 3 . 32x^5 + 64x^6$$

$$= 729 - 2916x + 4860x^2 - 4320x^3 + 2160x^4 - 576x^5 + 64x^6.$$

Example II. *Find the middle term of* $(2a-5x)^{13}$.

There are 13 terms, therefore the middle term is the 7^{th}.

The 7^{th} term $= +{}_{13}C_6 (2a)^6 (5x)^6$

$$= \frac{13 . 12 . 11 . 10 . 9 . 8 . 7}{1 . 2 . 3 . 4 . 5 . 6} \cdot 10^6 a^6 x^6$$

$$= 924000000\, a^6 x^6.$$

Example III. *Find the term, not involving x, in the expansion of* $\left(x^3 - \frac{3}{x^5}\right)^{15}$.

Method I. General term $= {}_{15}C_r \left(\frac{-3}{x^5}\right)^r (x^3)^{15-r}$

$$= (-3)^r {}_{15}C_r x^{30-8r}.$$

r is to be chosen so that $30-8r = 0$; therefore $r = 6$.

$$\text{Required term} = (-3)^6 \frac{\overset{5}{15} \cdot \overset{7}{14} \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6}}$$
$$= 3648645.$$

Method II.

$$\left(x^3 - \frac{3}{x^3}\right)^{15} = \frac{(x^5-3)^{15}}{x^{45}}.$$

We require, therefore, the coefficient of x^{45} in $(x^5-3)^{15}$;

i.e. the coefficient of $(x^5)^{15-6}$ in $(x^5-3)^{15}$;

i.e. the coefficient of the 7th term in $(x^5-3)^{15}$;

$\therefore$ required term $= {}_{15}C_6(-3)^6$;

and the work proceeds as in Method I.

Example IV. *Prove that the coefficient of the middle term of* $(1+x)^{2n}$ *is equal to the sum of the squares of the coefficients of all the terms in* $(1+x)^n$.

For convenience write c_r in place of ${}_nC_r$.

Then $(1+x)^n = 1 + c_1x + c_2x^2 + \ldots + c_2x^{n-2} + c_1x^{n-1} + x^n$; (A)

or $(1+x)^n = x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_2x^2 + c_1x + 1$; (B)

$\therefore$ $(1+x)^{2n}$ = the product of these two series.

Now the middle term of $(1+x)^{2n}$ is the $\overline{n+1}|^{\text{th}}$ term;

i.e. the term containing x^n.

Hence the middle term = term containing x^n in the product of the two series A and B;

$\therefore$ coefficient of middle term

$$= 1 + c_1^2 + c_2^2 + c_3^2 + \ldots + c_2^2 + c_1^2 + 1$$
$$= \text{sum of squares of coefficients in } (1+x)^n.$$

288. Example. *Find the numerically greatest coefficient in the expansion of* $(3-2x)^{13}$. *Also find the greatest term when* $x = 5$.

$$u_{r+1} = \frac{13(13-1)\ldots(13-r+2)(13-r+1)}{\lfloor r} 3^{13-r}(-2x)^r,$$

$$u_r = \frac{13(13-1)\ldots(13-r+2)}{\lfloor r-1} 3^{13-r+1}(-2x)^{r-1}.$$

$$\therefore\ u_{r+1} = u_r \times \frac{13-r+1}{r} \cdot \frac{-2x}{3}.$$

The x of the formula is here 3;

" a " " " " " " $-2x$.

Hence the $\overline{r+1}|^{\text{th}}$ coefficient is numerically greater than r^{th} coefficient,

so long as $\frac{13-r+1}{r}\cdot\frac{2}{3}>1$;

i.e. so long as $28-2r>3r$;

i.e. so long as $28>5r$;

i.e. so long as $5\frac{3}{5}>r$;

$\therefore$ the coefficient is greatest when $r=5$;

i.e. the 6^{th} term has the greatest coefficient.

Again $u_{r+1}>u_r$, (numerically)

so long as $\frac{13-r+1}{r}\cdot\frac{2x}{3}>1$;

i.e. so long as $\frac{14-r}{r}\cdot\frac{10}{3}>1$, when $x=5$;

i.e. so long as $140-10r>3r$;

i.e. so long as $140>13r$;

i.e. $10\frac{10}{13}>r$;

$\therefore$ the term is greatest when $r=10$;

i.e. the 11^{th} term is the greatest.

Examples XXVIII.

Expand:

1. $(1-x)^7$.
2. $(2x+3y^2)^5$.
3. $\left(\frac{x}{2}-2\right)^5$.
4. $(2a-b)^7$.
5. $\left(x+\frac{1}{x}\right)^5$.
6. $(a-\sqrt{3})^6$.
7. $\left(1-\frac{x}{2}\right)^{10}$.
8. $(a+2x)^8$.
9. $\left(5-\frac{x}{6}\right)^6$.
10. Find the middle term in the expansion of $\left(\frac{2a}{3}-\frac{3b}{4}\right)^{10}$.
11. Find the coefficient of x^3 in $(32x-\frac{1}{2})^{20}$.
12. Find the coefficient of x^5 in $\left(2-\frac{x}{3}\right)^9$.
13. What is the coefficient of x^9 in $(2+\frac{3}{4}x)^{17}$?
14. Find the coefficient of x^5 in $(1-x+x^2)(1+x)^{11}$.
15. What is the coefficient of x^{2r+1} in $\left(x-\frac{1}{x}\right)^{2n+1}$?
16. Write down the coefficient of x^n in the expansion of $(1-x^{\frac{1}{2}})^{48}$, where n is a positive integer.

17. Find the coefficient of x^{17} in the expansion of $\left(\frac{x^2}{2}-\frac{2}{x}\right)^{16}$.

18. If $(1+x)^n = a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n$, find the value of $a_0a_n+a_1a_{n-1}+a_2a_{n-2}+\ldots+a_{n-1}a_1+a_na_0$.

19. Prove that the sum of the coefficients of the r^{th} and $(r+1)^{\text{th}}$ terms of the expansion of $(1+x)^n$ is equal to the coefficient of the $(r+1)^{\text{th}}$ term of the expansion of $(1+x)^{n+1}$, n being an integer.

20. If $(1+x)^n = c_0+c_1x+c_2x^2+\ldots+c_{n-1}x^{n-1}+c_nx^n$ where n is a positive integer, prove that

(i) $c_0+c_1+c_2+\ldots+c_{n-1}+c_n = 2^n$;

(ii) $c_r = c_{n-r}$;

(iii) $c_0c_1+c_1c_2+c_2c_3+\ldots+c_{n-1}c_n = \frac{\lfloor 2n}{\lfloor n+1 \rfloor n-1}$.

21. Write down the coefficient x^9 in $(5a^3-4x^3)^7$; and find the numerically greatest coefficient if $a=2$.

22. Find the middle terms in the expansions of $(a-x)^{10}$ and $\left(x-\frac{1}{x}\right)^{4n}$.

23. Show that the $(r+1)^{\text{th}}$ term in the expansion of $\left(1+\frac{x}{m}\right)^m$ is $\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\ldots\left(1-\frac{r-1}{m}\right)\frac{x^r}{\lfloor r}$.

Hence show that if m is a positive integer,

$$\left(1+\frac{1}{m+1}\right)^{m+1} > \left(1+\frac{1}{m}\right)^m.$$

24. Find the term independent of x in the expansion of

$$\left(3x-\frac{2}{x^2}\right)^{15}.$$

25. Find the coefficient of x^4 in $(1-x+x^2)^8$.

26. Expand $(1+x+x^2)^7$ as far as x^6.

27. Find the coefficient of x^4 in the expansion of

$$(1+2x-3x^2+x^3)^5.$$

28. Find the value of $(1+x)^6+(1-x)^6$ when $x=\sqrt{3}$.

29. Find the value of the greatest term in $(2-3x)^5$ when $x=\frac{1}{3}$. Find also the value of the whole expansion.

30. Find the value of the term independent of x in

$$\left(x^3-2+\frac{1}{x^3}\right)^6.$$

CHAPTER XXIX*

INEQUALITIES. LIMITING VALUES. CONVERGENCY AND DIVERGENCY OF SERIES.

Inequalities.

289. **Signs of inequality.** Instead of the negative signs $\gtrless$ and $\lessgtr$ it is now the custom to use the positive signs $\leq$ and $\geq$, which are read 'is less than or equal to', 'is greater than or equal to' respectively.

290. If $a-b$ is positive, then a is said to be greater than b. Here either a or b or both may be negative; and some care is necessary in dealing with inequalities when a and b are not restricted to being positive.

The following propositions are easily proved and their proofs are left as exercises for the student.

If a and b are real quantities such that $a-b>0$,

(i) $a+x>b+x$;

(ii) $ax>bx$, when x is positive;

(iii) $ax<bx$, when x is negative; this includes $-a<-b$;

(iv) $a^n>b^n$, when n, a, b are all positive.

(v) $\frac{1}{a}<\frac{1}{b}$, when a and b are of the same sign;

(vi) $a^n<b^n$, when n is negative and a and b are of the same sign.

(vii) ax need not be greater than by, even though $a>b$ and $x>y$.

It is assumed in the remainder of this chapter that all letters denote positive quantities unless it is explicitly stated that their values are unrestricted.

291. If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \frac{a_n}{b_n}$ are fractions with positive denominators, then the fraction $\frac{a_1+a_2+a_3 \ldots +a_n}{b_1+b_2+b_3 \ldots +b_n}$ is intermediate in value between the greatest and least of them.

* I am indebted to Dr. F. S. Macaulay, St. Paul's School, for much valuable advice in connexion with this chapter.—W. E. P.

Let $\frac{a_r}{b_r}$ be the greatest of the fractions and its value k.

$$\frac{a_1}{b_1} < k, \quad \therefore \quad a_1 < b_1 k, \text{ since } b_1 \text{ is positive;}$$

similarly,
$$a_2 < b_2 k$$
$$a_r = b_r k$$
$$a_n < b_n k$$

Add $$(a_1 + a + \ldots + a_n) < (b_1 + b_2 + \ldots + b_n) k,$$

i.e. $$\frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} < k < \frac{a_r}{b_r}$$ the greatest fraction.

Similarly, $\frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n}$ can be shown to be greater than the least of the fractions.

If the denominators are all negative, a similar proof holds.

Corollary. If $k_1, k_2, \ldots k_n$ are any quantities of the same sign, then $\frac{k_1 a_1 + k_2 a_2 + k_3 a_3 + \ldots + k_n a_n}{k_1 b_1 + k_2 b_2 + k_3 b_3 + \ldots + k_n b_n}$ is intermediate in value between the greatest and least of the fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots \frac{a_n}{b_n}$.

292. In dealing with real quantities, any square quantity must be positive or zero. This fact leads to the proof of many inequalities.

293. If a and b are real, positive or negative, then $a^2 + b^2 \geq 2ab$.

For $$(a-b)^2 \geq 0,$$
$$\therefore \quad a^2 + b^2 \geq 2ab.$$

294. If $x + y = c$, then xy is greatest when $x = y$, i.e. $xy \leq \frac{c^2}{4}$.

For $$xy = x(c-x) = \frac{c^2}{4} - \left(\frac{c}{2} - x\right)^2;$$

$\therefore$ xy is greatest when $\left(\frac{c}{2} - x\right)^2$ is least, i.e. when $\frac{c}{2} - x = 0$, i.e. when $x = y$;

and the maximum value is $\frac{c^2}{4}$.

295. **The Arithmetic mean of two real different quantities a and b is greater than the Geometric mean.**

For $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2}$ = a positive quantity.

$\therefore\ A > G.$

positive

296. **If there are n quantities, their sum divided by n is called their Arithmetic mean; and the n^{th} root of their product is called their Geometric mean.**

The Arithmetic mean of n quantities $a, b, c, d \ldots$ which are not all equal is greater than their Geometric mean.

In the product $abcd\ldots$, suppose a and b change their values their sum always remaining the same. Then the product ab is greatest when $a = b$. Hence if two factors of the product $abcd\ldots$ are not equal, the product can be increased by making those two factors equal, their sum remaining unaltered. It follows that the greatest value of the product is obtained by making all the factors equal, the sum remaining $a+b+c+d+\ldots$.

Each factor is then $\frac{a+b+c+d+\ldots}{n}$,

so that the maximum value of the product is

$$\left(\frac{a+b+c+d+\ldots}{n}\right)^n;$$

i.e. $$\left(\frac{a+b+c+d+\ldots}{n}\right)^n > abcd\ldots;$$

or $$\frac{a+b+c+d+\ldots}{n} > \sqrt[n]{abcd\ldots},$$

i.e. A. mean > G. mean.

If we allow for the possibility that $a, b, c, \ldots$ may be all equal, we have the more general statement

A. mean $\geq$ G. mean.

297. **Example I.** *If $x > y$, show that*

$$\frac{x^4-y^4}{4y^3} > x-y > \frac{x^4-y^4}{4x^3}$$ *if x and y are both positive.*

Since $x > y$ and x and y are not negative,

$\therefore\ \frac{x}{y} > 1$ and $\frac{y}{x} < 1.$

Now $x^4-y^4=(x-y)(x^3+x^2y+xy^2+y^3)$,

$$\therefore \quad \frac{x^4-y^4}{4y^3}=(x-y)\frac{\left(\frac{x}{y}\right)^3+\left(\frac{x}{y}\right)^2+\left(\frac{x}{y}\right)+1}{4};$$

but $\frac{x}{y}>1$, $\quad \therefore \quad \left(\frac{x}{y}\right)^3+\left(\frac{x}{y}\right)^2+\left(\frac{x}{y}\right)+1>4.$

Hence $\quad \frac{x^4-y^4}{4y^3}>x-y.$

Similarly, $\frac{x^4-y^4}{4x^3}=(x-y)\frac{1+\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2+\left(\frac{y}{x}\right)^3}{4}$

$$<x-y \text{ since } \frac{y}{x}<1.$$

Hence $\quad \frac{x^4-y^4}{4y^3}>x-y>\frac{x^4-y^4}{4x^3}.$

Note. If x and y are both negative, $\frac{x}{y}<1$ and $\frac{y}{x}>1$. In this case, therefore, the signs of inequality are reversed.

Example II. *Prove that* $(a^2+b^2+c^2)>(bc+ca+ab)$, *provided* a, b, c *are real quantities and are not all equal.*

Since a, b, c are real, $(b-c)$, $(c-a)$, $(a-b)$ are all real;

$$\therefore \quad (b-c)^2+(c-a)^2+(a-b)^2>0,$$

i.e. $\quad 2(a^2+b^2+c^2)-2(bc+ca+ab)>0,$

i.e. $\quad (a^2+b^2+c^2)>(bc+ca+ab).$

Example III. *Prove that* $(a+b+c)^3\geq 27abc.$

The A. mean of $a, b, c \geq$ the G. mean,

i.e. $\quad \frac{a+b+c}{3}\geq \sqrt[3]{abc}.$

Hence $\quad (a+b+c)^3\geq 27abc.$

Examples XXIX a.

Prove that in any inequality:

1. Any term may be transposed from one side to the other provided its sign be changed.

2. Both sides may be multiplied or divided by the same positive quantity.

3. The sides may be transposed, if the sign of inequality is at the same time reversed.

4. If each side is inverted, the sign of inequality must be reversed.

5. If all the signs be changed, the sign of inequality must be reversed.

6. Where would the proof of § 291 break down if the denominators were of different signs?

7. Write out the proof of § 291, when the denominators are all negative.

8. If $a > b$ and $c > d$ and are not necessarily all positive, is it necessarily true that $ac > bd$? Illustrate by numerical examples.

9. If a and b are any quantities such that $a - b > 0$, show that a^n is not necessarily greater than b^n even though n be positive.

10. Show that the sum of any positive quantity and its reciprocal cannot be less than two. State the corresponding theorem for a negative quantity.

11. Show that $\frac{n}{(n+1)(n+3)} > \frac{n+1}{(n+2)(n+4)}$.

12. If a, b, c be different positive numbers, show that

$$bc(b+c) + ca(c+a) + ab(a+b) > 6abc.$$

13. Prove that four times the sum of the cubes of any two different numbers is greater than the cube of the sum of the two numbers.

14. Show that $a^3 + b^3 + c^3 \geq 3abc$, provided $a + b + c > 0$.

15. Show that $2(a^3 + b^3 + c^3) \geq bc(b+c) + ca(c+a) + ab(a+b)$, if a, b, c are positive.

16. Prove that $(a+b+c)^3 \leq 9(a^3 + b^3 + c^3)$, if a, b, c are positive.

17. Prove that $abc \geq (b+c-a)(c+a-b)(a+b-c)$ if a, b, c are positive.

18. Prove that $abc \geq (2a-b)(2b-c)(2c-a)$ if a, b, c are positive.

298. Small quantities.

Example. *Find the value, correct to* 3 *decimal places, of* $\frac{1}{1-x}$ *when* x *equals* (i) ·1, (ii) ·001, (iii) ·00001.

In all three cases it is found by division that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots .$$

Case I. $x = .1$.

Then $x^2 = .01$, $x^3 = .001$, $x^4 = .0001$.

For the required degree of accuracy x^4 and higher powers may be neglected.

Hence $\frac{1}{1-x} = 1+x+x^2+x^3 = 1.111$.

Case II. $x = .01$.

Then $x^3 = .0001$; therefore x^2 and higher powers may be neglected.

Hence $\frac{1}{1-x} = 1+x = 1.010$.

Case III. $x = .00001$.

Here x and all higher powers may be neglected.

Hence $\frac{1}{1-x} = 1 = 1.000$.

The degree of smallness of a quantity is often expressed by mentioning the lowest power that may be neglected, e. g. if x is a small quantity whose square may be neglected, then $\frac{1}{1+x} = 1-x$. To say that $x = 0$ is equivalent to saying that its first power may be neglected.

If x and y are small quantities such that their squares may be neglected, their product also may be neglected; thus, in such cases, $(1+x)(1+y) = 1+x+y$.

LIMITING VALUES

299. Example. *Determine the value of* $\frac{x^2-1}{x-1}$ *when* $x = 1$.

Substituting, we have

$$\frac{x^2-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}.$$

Since $0 \times x = 0$ whatever finite value x may have;

$\therefore \frac{0}{0} = x$ where x may have any finite value. This is usually expressed by saying that $\frac{0}{0}$ is indeterminate.

If $x-1$ is not nothing, we may divide by it; hence when $x \neq 1$, $\frac{x^2-1}{x-1} = x+1$.

If $x = 1.01$, fraction $= 2.01$

$x = 1.00001$, fraction $= 2.00001$

$x = 1-.01$, fraction $= 2-.01$.

$x = 1-.00001$, fraction $= 2-.00001$.

As x approaches more nearly to the value 1, the fraction approaches more nearly to the value 2; and the fraction may

be made to differ from the value 2 by as small a quantity as we please by taking x sufficiently close to the value 1; i. e. the limiting value of $\frac{x^2-1}{x-1}$, as x approaches 1, is 2.

300. If $f(x)$ approaches the value b when x approaches the value a, and can be made to differ from b by less than any assigned quantity by taking x sufficiently nearly equal to a, then b is called the limiting value or limit of $f(x)$ as x approaches a. This is written

$$\mathop{\mathrm{L}}_{x \to a} f(x) = b.^*$$

Alternative definition. If $f(x)$ is a function of x which has the value $u+\epsilon$, when $x = a+h$; and if by making h indefinitely small, ϵ may be made less than any assignable quantity, then u is called the limiting value of $f(x)$ when x approaches a.

301. The symbols 0 and ∞.

In Arithmetic and more elementary Mathematics the symbol 0 stands for the absolute zero, i. e. the difference between two quantities that are exactly equal. In more advanced work it may also stand for the limit of a variable quantity which diminishes indefinitely, just as ∞ stands for the limit of a variable quantity which increases indefinitely. It is important to bear in mind that ∞ does not stand for a definite number, but for any number that is larger than any conceivable number; hence it is not necessarily true that $\infty = \infty$.

A quantity which is not infinitely large is said to be finite.

302. **Fundamental Theorems on Limits.**

Theorem I. If the limits, when x approaches a, of a finite number n of functions of x, viz. $f_1, f_2, f_3, \ldots f_n$ are each nothing, then the limit of their sum is also nothing.

* The symbol $x \to a$ (x approaches a) is of very recent introduction. The older and, at present, more usual notation is $\mathop{\mathrm{L}}_{x=a} f(x)$. This is misleading for x does not actually equal a. In some of the examples,

the older notation and corresponding wording (when $x = a$) are retained in many of the examples of this book.

For suppose that, when $x = a+h$, the functions are respectively $h_1, h_2, \dots h_n$. Let h_r be the greatest of the n quantities $h_1, h_2, \dots h_n$.

Then $$\Sigma(f_r) = h_1+h_2+\dots+h_n, \text{ when } x = a+h.$$
$$< nh_r.$$

Let E be an infinitely small quantity; then, by the definition of limit, h can be chosen so that h_r is less than $\frac{E}{n}$. Hence h can be so chosen, that $\Sigma(f_r) < E$, i.e. less than any assigned quantity.

Therefore $$\mathop{\mathrm{L}}_{x \to a} (f_1+f_2+\dots+f_n) = 0.$$

Theorem II. If $u_1, u_2, \dots u_n$ are the respective limits of $f_1, f_2, \dots f_n$ when x approaches a then $\Sigma(u_r)$ is the limit of $\Sigma(f_r)$.

When $x = a+h$, let $u_1+h_1, u_2+h_2 \dots$ be the respective values of the functions;

$$\Sigma(f_r) = u_1+h_1+u_2+h_2\dots, \text{ when } x = a+h$$
$$= \Sigma(u_r)+h_1+h_2+\dots+h_n.$$

h may be taken so small, that $h_1, h_2, \dots h_n$ are all infinitely small.

Therefore $$\mathop{\mathrm{L}}_{x \to a} \Sigma(f_r) = \Sigma(u_r).$$

Theorem III. The limit of the product of a finite number n of functions is the product of their limits, provided the limits are none of them infinite.

$$f_1 f_2 \dots f_n = (u_1+h_1)(u_2+h_2)(u_3+h_3)\dots(u_n+h_n), \text{ when } x = a+h$$
$$= u_1 u_2 \dots u_n + \Sigma(h_1 u_2 u_3 \dots u_n)$$
$$+ \Sigma(h_1 h_2 u_3 u_4 \dots u_n) + \dots.$$

Now h may be taken so small that all products containing one of the quantities $h_1, h_2, \dots$ as factors can be neglected unless one or more of the quantities $u_1 u_2 \dots u_n$ is infinite.

Hence, if $u_1, u_2, \dots u_n$ are all finite,

$$\mathop{\mathrm{L}} f_1 f_2 \dots f_n = u_1 u_2 \dots u_n.$$

If n is infinite, the theorem is not necessarily true; for $\Sigma(h_1 u_2 u_3 \dots u_n)$ may be finite. An important limit of this type is given in § 328, p. 482.

Theorem IV. The limit of the quotient of two functions is the quotient of their limits, provided the limit of the divisor is not zero and both limits are not infinite.

With the notation of Theorem II.

$$\frac{f_1}{f_2} = \frac{u_1+h_1}{u_2+h_2} \text{ when } x = a+h$$

$$= \frac{u_1}{u_2} + \frac{u_1+h_1}{u_2+h_2} - \frac{u_1}{u_2} = \frac{u_1}{u_2} + \frac{u_2 h_1 - u_1 h_2}{u_2(u_2+h_2)}.$$

If u_2 is not infinite the numerator of the second fraction can be made as small as we please, but the denominator is by hypothesis not zero, therefore the second fraction can be made as small as we please.

$$\therefore \quad \lim \frac{f_1}{f_2} = \frac{u_1}{u_2}.$$

If u_2 is infinite and u_1 is not,

$$\frac{f_1}{f_2} = \frac{u_1}{u_2} + \frac{h_1 - \frac{u_1}{u_2} h_2}{u_2+h_2} = 0 \left(\text{since } \frac{u_1}{u_2} \text{ is indefinitely small}\right)$$

and the theorem is true.

If u_1 u_2 are both zero or both infinite each case must be specially investigated.

303. The following limits are continually required

(i) $\lim_{x \to 0} \frac{a}{x} = \infty$, (ii) $\lim_{x \to \infty} \frac{a}{x} = 0$,

(iii) $\lim_{n \to \infty} x^n = \infty$ when $x > 1$, (iv) $\lim_{n \to \infty} x^n = 0$ when $x < 1$.

These hardly require proof; but proofs can be given as follows:

(i) $\frac{a}{x}$ can be made as large as we please by taking x small enough, provided a is not zero.

$$\therefore \quad \lim_{x \to 0} \frac{a}{x} = \infty.$$

(ii) $\frac{a}{x}$ can be made as small as we please by taking x large enough, provided a is not infinite.

$$\therefore \quad \lim_{x \to \infty} \frac{a}{x} = 0.$$

(iii) Since $x > 1$, $x = 1+c$ where c is positive.

By the Binomial Theorem $(1+c)^n = 1+nc+\frac{n(n-1)}{1.2}c^2+\ldots.$

$$\therefore \quad x^n > 1+nc.$$

However small a constant c may be, nc can be made as large as we please by taking n large enough.

$$\therefore \quad \lim_{n\to\infty} x^n = \infty \text{ when } x > 1.$$

It follows that

$$\lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r} \text{ when } r < 1.$$

(iv) Let $y = \frac{1}{x}$, then $y > 1$.

$$\therefore \quad \lim_{n\to\infty} y^n = \infty \text{ (by iii)};$$

also
$$\lim_{n\to\infty} x^n = \lim_{n\to\infty} \frac{1}{y^n}.$$

$$\therefore \quad \lim_{n\to\infty} x^n = 0 \text{ when } x < 1 \text{ (by ii and iii)}.$$

304. Theorem I. The limit of the infinite series

$$a_0 + a_1x + a_2x^2 + \ldots,$$

where all the coefficients are finite quantities, is a_0 when x is indefinitely diminished.

Let S be the sum of all the terms following the first and let a_k be the greatest of the coefficients following a_0.

Then
$$S < a_k(x + x^2 + x^3 \ldots)$$
$$< a_k\, x\frac{(1-x^n)}{1-x};$$

$$\therefore \quad \lim_{x\to 0}(S) < a_k \lim_{x\to 0} x \times \lim_{x\to 0}(1-x^n) \times \lim_{x\to 0}\frac{1}{1-x}$$

(Theorem III, § 302.)

$$< a_k \times 0 \times 1 \times 1 \text{ (where 0 represents an infinitely small quantity)};$$

i.e. $\lim_{x\to 0}(S) = 0;$

i.e. Sum of series $= a_0$ when x is indefinitely diminished.

Theorem II. In a series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

where all the coefficients are finite quantities, any term may be made infinitely great compared with the sum of all the following terms by taking x sufficiently small; and may be made infinitely great compared with the sum of all the preceding terms by taking x sufficiently large.

Take $a_r x^r$ as the particular term and let a_n be the greatest of all the following coefficients, and a_k the greatest of all the preceding coefficients.

Sum of all the following terms $< a_n (x^{r+1} + x^{r+2} + \ldots \text{ to } \infty)$.

x may be as small as we please and may therefore be less than one.

$\therefore$ Sum of all the following terms $< \dfrac{a_n x^{r+1}}{1-x}$;

$\therefore$ Ratio of $a_r x^r$ to sum of following terms $> a_r x^r \div \dfrac{a_n x^{r+1}}{1-x}$

$$> \frac{a_r}{a_n} \times \frac{1-x}{x}$$

$$> \frac{a_r}{a_n}\left(\frac{1}{x} - 1\right).$$

Now $\dfrac{1}{x}$ may be made as large as we please.

$\therefore$ $\dfrac{1}{x} - 1$ may be made as large as we please.

$\therefore$ $\dfrac{a_r}{a_n}\left(\dfrac{1}{x} - 1\right)$ may be made as large as we please.

Hence the first part of the theorem is proved.

Similarly, the ratio of $a_r x^r$ to sum of preceding terms

$$> a_r x^r \div \frac{a_k (x^r - 1)}{x - 1}$$

$$> \frac{a_r}{a_k} (x-1) \frac{x^r}{x^r - 1} > \frac{a_r}{a_k} (x-1), \text{ i.e. } > \text{any assigned quantity.}$$

305. If all the coefficients are finite, the limit of the fraction

$$\frac{a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots b_n x^n}$$

is $\dfrac{a_0}{b_0}$ when x is indefinitely diminished and $\dfrac{a_n}{b_n}$ when x is indefinitely increased.

For by § 304 the numerator $= a_0 + S_1$, and the denominator $= b_0 + S_2$ where S_1 and S_2 are indefinitely small when x is indefinitely diminished.

$\therefore$ Limit of the fraction $= \lim_{x \to 0} \dfrac{a_0 + S_1}{b_0 + S_2} = \dfrac{a_0}{b_0}$. (Theorem IV, § 302.)

Again,

$$\text{Fraction} = \frac{a_n + a_{n-1}y + a_{n-2}y^2 \ldots + a_0 y^n}{b_n + b_{n-1}y + b_{n-2}y^2 \ldots + b_0 y^n} \text{ where } y = \frac{1}{x}$$

$$= \frac{a_n}{b_n} \text{ when } y \text{ is indefinitely diminished}$$

$$= \frac{a_n}{b_n} \text{ when } x \text{ is indefinitely increased.}$$

Hence in finding the limiting values of fractions we need consider only the lowest powers in the numerator and denominator when the variable diminishes without limit, and only the highest powers when the variable increases without limit.

306. Indeterminate quantities. Quantities are said to be indeterminate when they reduce to one of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$.

A few examples will show the algebraical method of dealing with them.

Example I. *Find the value of* $\dfrac{2x^2 - x - 1}{3x^2 - 5x + 2}$ *when* $x = 1$.

This is of the form $\frac{0}{0}$.

Put $x = 1 + h$ where h is small but is not zero.

$$\text{Fraction} = \frac{2(1 + 2h + h^2) - (1 + h) - 1}{3(1 + 2h + h^2) - 5(1 + h) + 2}$$

$$= \frac{3h + 2h^2}{h + 3h^2}$$

$$= \frac{3 + 2h}{1 + 3h} \text{ since } h \text{ is not nothing.}$$

This can be made to differ from $\frac{3}{1}$ by as little as we please by taking h small enough.

Hence $$\lim_{x \to 1} \frac{2x^2 - x - 1}{3x^2 - 5x + 2} = 3.$$

This question might also be worded '*Evaluate* $\dfrac{2x^2 - x - 1}{3x^2 - 5x + 2}$ *when* $x = 1$'.

Example II. *Find the value of* $\frac{n(n+1)}{2n^2}$ *when* $n = \infty$.

This is of the form $\frac{\infty}{\infty}$.

$$\text{Fraction} = \frac{n^2+n}{2n^2}$$
$$= \frac{1}{2} + \frac{1}{2n}.$$

This can be made to differ from $\frac{1}{2}$ by as little as we please by taking n large enough.

Hence $$\lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}.$$

Example III. *Find the value of* $\left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} \dots + \frac{n}{n^2}\right)$ *when* $n = \infty$.

Each term in the bracket becomes 0 when $n = \infty$; and there are an infinite number of terms; hence the expression is of the form $0 \times \infty$.

$$\text{Expression} = \frac{1+2+3 \dots +n}{n^2}$$
$$= \frac{n(n+1)}{2n^2},$$

and it has been shown in the previous example that by increasing n we can make this differ by as little as we please from $\frac{1}{2}$.

Hence $$\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} \dots + \frac{n}{n^2}\right) = \frac{1}{2}.$$

307. Graphical Illustration of $\frac{0}{0}$.

Let (x_1, y_1), (x_2, y_2) be the coordinates of two points P_1 and P_2; and let $y = mx + b$ be the equation of the line joining them. By substituting the coordinates (x_1, y_1), (x_2, y_2) and eliminating m and b, we find that this equation is Fine & Thompson, §20. Loney, §62. Askwith, Ch III.

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1).$$

If we suppose P_2 to move up to P_1 so that ultimately $x_2 = x_1$ and $y_2 = y_1$, the equation becomes

$$y - y_1 = \frac{0}{0}(x - x_1),$$

where the coefficient of x is indeterminate and may be any number.

This is what we should expect, for if the points actually coincide any line through P_1 passes through both of them, and the equation of any line passing through (x_1, y_1) is of the form $y-y_1 = m(x-x_1)$ where m is any number.

If, however, P_1 and P_2 lie on some curve, the doubt as to the value of the coefficient of x may be removed since the coefficient approaches a definite limiting value as P_2 moves up indefinitely close to P_1.

Suppose P_1 and P_2 lie on the circle whose equation is

$$x^2+y^2 = a^2,$$

then
$$x_1^2+y_1^2 = a^2,$$
$$x_2^2+y_2^2 = a^2;$$

$\therefore$
$$y_1^2-y_2^2 = -(x_1^2-x_2^2).$$

So long as P_1 and P_2 have not actually coincided we may divide by x_1-x_2.

$\therefore$
$$\frac{y_1-y_2}{x_1-x_2} = -\frac{x_1+x_2}{y_1+y_2}.$$

Hence, when P_2 approaches indefinitely close to P_1, $\frac{y_1-y_2}{x_1-x_2}$ approaches the limiting value $-\frac{x_1}{y_1}$.

But, when P_2 moves along the circle indefinitely close to P_1, the line P_1P_2 approaches indefinitely near to a definite fixed position which is called the tangent to the circle at P_1.

By comparing the Algebraical and Geometrical limits we see that the equation of the tangent at P_1 is $y-y_1 = -\frac{x_1}{y_1}(x-x_1)$.

This reduces to $xx_1+yy_1 = a^2$.

The same process can be applied when P_1, P_2 lie on any other curve whose equation is given.

Examples XXIX b.

1. Prove that $\lim_{h=0} \frac{(x+h)^3-x^3}{h} = 3x^2$.

2. Prove that $\lim_{h=0} \frac{(x+h)^n-x^n}{h} = nx^{n-1}$ when n is a positive integer.

3. Find the value of $\underset{n=\infty}{\mathrm{L}} \frac{1^2+2^2+3^2\ldots+n^2}{n^3}$.

4. Prove (i) $\underset{n\to\infty}{\mathrm{L}} \frac{x^n}{n} = \infty$ when $x>1$, (ii) $\underset{x\to\infty}{\mathrm{L}} (nx^n) = 0$ when $x<1$.

5. Find the limiting value of $\frac{x^n}{\underline{n}}$ when n increases indefinitely and x is finite.

6. Find the value of

$$\underset{n\to\infty}{\mathrm{L}} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r}{n}\right)}{\underline{r}}$$ when r is finite.

What difference will it make if r is also infinite but less than n?

Evaluate the following fractions (see Example I, § 306):

7. $\frac{x^2-4x+3}{x^2+7x-8}$ when $x=1$.

8. $\frac{2x^3-5x^2-4x+12}{x^3-12x+16}$ when $x=2$.

9. $\frac{3x^5-4x^3+x^2-7}{4x^5-7x^3+3x+2}$ when $x=0$ and when $x=\infty$.

10. Find the equation of the tangent to the parabola $y=x^2$ at the point x_1, y_1.

Convergency and Divergency of series.

308. By division it can be shown that

$$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots \text{ to infinity.}$$

Put $x=10$;

$$\frac{1}{1-x} = \frac{1}{1-10} = -\frac{1}{9},$$

and the series $= 1+10+100+1000+10000+\ldots$ to infinity.

In this case the series is not equal to the fraction; the more terms that are taken the more the series diverges from the value

of the fraction, and the sum of the series tends to an infinite value. Such a series is **Divergent.**

Put $x = \cdot 1$,

then $$\frac{1}{1-x} = \frac{1}{1-\cdot 1} = 1\frac{1}{9},$$

and the series $= 1 + \cdot 1 + \cdot 01 + \cdot 001 \ldots$ to infinity

$= 1 \cdot 111111 \ldots$ to infinity.

Here the more terms of the series that are taken the nearer the series converges to the value of the fraction, and the sum of the series tends to the limiting value $1\frac{1}{9}$ which is the value of the fraction. Such a series is **Convergent.**

Put $x = -1$,

then $$\frac{1}{1-x} = \frac{1}{2},$$

and the series $= 1 - 1 + 1 - 1 + 1 - 1 \ldots$ to infinity

$= 0$ if infinity is even, but $= 1$ if infinity is odd.

This series is said to **Oscillate.**

We now see that before we substitute a numerical value for x in any series we ought to determine whether the series is convergent for that value of x.

309. An infinite series is **Convergent** if the sum of n terms approaches a definite finite value and can be made to differ from that finite value by less than any assigned quantity by taking n sufficiently large.

This is expressed more concisely:—

An infinite series is **Convergent** if the sum of n terms approaches a definite finite limit when n is indefinitely increased.

A series is **Divergent** if the limit of the sum of n terms is infinite.

It may happen that the sum of n terms tends to different limits according to the form n is supposed to have (e. g. the limit may have one value when the infinite value of n is supposed odd and another when it is supposed even); such a series is said to be **Oscillating.**

The Geometric series supplies examples of each of these.

Consider the series $a+ax+ax^2+\dots$ to ∞,

$$S_n = a\frac{1-x^n}{1-x} = \frac{a}{1-x} - \frac{ax^n}{1-x}.$$

If x is numerically less than 1, $\lim_{n\to\infty} \frac{ax^n}{1-x} = 0$.

$\therefore\ S_\infty = \frac{a}{1-x}$, and the series is convergent.

If x is numerically greater than 1, $\lim_{n\to\infty} \frac{ax^n}{1-x} = \infty$.

$\therefore\ S_\infty = \infty$, and the series is divergent.

If $x = 1$, $S_\infty = a+a+a+\dots$ to $\infty = \infty$,

and the series is divergent.

If $x = -1$, $S_\infty = a-a+a-a+a-a\dots$.

If ∞ is odd, $S_\infty = a$; if ∞ is even, $S_\infty = 0$.

Hence the series oscillates.

TESTS FOR CONVERGENCY

310. Whenever a formula can be found for S_n, the convergency or divergency of the series can be determined by finding the limit of S_n when $n = \infty$. When no formula can be found, the following tests are frequently used.

Theorem I. If a series $u_1+u_2+\dots+u_n+\dots$ is convergent, then $\lim_{n\to\infty} u_n$ must be zero.

For if the series is convergent,

$\mathrm{L}_{n\to\infty} S_{n-1}$ and $\mathrm{L}_{n\to\infty} S_n$ equal the same finite quantity;

i.e. $\mathrm{L}_{n\to\infty} S_n - \mathrm{L}_{n\to\infty} S_{n-1} = 0$;

i.e. $u_n = 0$ when n is infinite.

The converse of this proposition is not necessarily true as is shown by Ex. II, § 311.

Theorem II. An infinite series of positive quantities is convergent if its sum can be proved to be less than a definite finite quantity.

This may perhaps be regarded as axiomatic: the proof is difficult, and we do not propose giving it.

Example. *Consider the series*

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots$ to infinity.

$\mathrm{L}_{n\to\infty} u_n = \mathrm{L}_{n\to\infty} \frac{1}{n} = 0$; $\therefore$ series may be convergent.

$S \equiv (1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})$ . . . . . (i)

$S \equiv 1-(\frac{1}{2}-\frac{1}{3})-(\frac{1}{4}-\frac{1}{5})$ . . . . . (ii)

Form I shows that the series equals the sum of a number of positive quantities since $\frac{1}{n}-\frac{1}{n+1}$ is positive.

Form II shows that the sum is less than 1.

Hence the series is convergent.

Theorem III. If after a finite number of terms a series is of the form $u_1-u_2+u_3-u_4\ldots$ to infinity, where $u_1, u_2, \ldots u_n$ are positive quantities continually decreasing to the limit zero, then the series is convergent.

S = a finite quantity $+(u_1-u_2)+(u_3-u_4)+\ldots$

= a finite quantity + an infinite series of positive quantities,

also S = a finite quantity $+u_1-(u_2-u_3)-(u_4-u_5)\ldots$;

i.e. $S <$ a finite quantity.

Hence, by Theorem II, the series is convergent.

Theorem IV. If a series

$$v_1+v_2+v_3+\dots \text{ to infinity}$$

is known to be convergent; and if another series

$$u_1+u_2+u_3+\dots \text{ to infinity}$$

is such that $u_n < v_n$, then the second series is convergent.

For $\Sigma u < \Sigma v <$ a finite quantity. $\therefore$ The u series is also convergent.

Note. It follows from this theorem that if

$u_1+u_2+u_3\dots+u_n+\dots$ is convergent

then $u_1-u_2+u_3\dots+(-1)^{n+1}u_n+\dots$ is also convergent.

Similarly, if the v-series is divergent and $u_n > v_n$, then the u-series is also divergent.

Example. *Consider the series*

$$1+\frac{1}{1.2}+\frac{1}{1.2.3}+\dots\frac{1}{\lfloor n}+\dots \text{ to infinity.}$$

This series is less than the series

$$1+\frac{1}{2}+\frac{1}{2.2}+\dots\frac{1}{2^{n-1}}+\dots \text{ to infinity,}$$

for $$2^{n-1} = 1.2.2.2\dots 2$$
$$< 1.2.3.4\dots n.$$

$$\therefore \quad \frac{1}{2^{n-1}} > \frac{1}{\lfloor n},$$

i.e. each term of the first series after the second is less than the corresponding term of the second series, and the first and second terms of the series are the same.

Hence the limit of the sum of the given series

$<$ the limit of the sum of the second series,

$$< \frac{1}{1-\frac{1}{2}} < 2.$$

$\therefore$ The series is convergent.

Theorem V. If after a certain term the ratio of each term to the preceding term is less than a quantity k which is itself less than 1, the series is convergent.

Let series $= S_n+u_{n+1}+u_{n+2}+\dots$ to infinity;

where S_n is sum of a finite number of terms and is finite.

$$\frac{u_{n+2}}{u_{n+1}} < k; \qquad \therefore \quad u_{n+2} < u_{n+1}\,k.$$

Similarly, $u_{n+3} < u_{n+2}\,k < u_{n+1}\,k^2$.

&c.

$\therefore$ Series $< S_n + u_{n+1}\{1 + k + k^2 + \ldots \text{ to infinity}\}$

$< S_n + \dfrac{u_{n+1}}{1-k}$ since $k < 1$

$<$ a finite quantity.

Similarly, if the ratio is greater than a quantity greater than 1, the series is divergent.

It is usual to apply this test in another form which we give as Theorem VI.

Theorem VI. The series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ to infinity is convergent or divergent according as the limit of the ratio $\dfrac{u_{n+1}}{u_n}$ when n is infinite is numerically less or greater than one.

Let k be the limiting value of the ratio $\dfrac{u_{n+1}}{u_n}$.

Suppose k less than 1; and let a be a quantity greater than k but less than 1.

Then as the ratio $\dfrac{u_{n+1}}{u_n}$ continually approaches the value k it will at length be permanently less than a. That is, there will at length be a value n such that after the n^{th} term the ratio of each term to the succeeding term is always less than a which is less than 1. Therefore by Theorem V the series is convergent.

If k is greater than 1 we take a quantity less than k but greater than 1, and by a similar argument the series is shown to be divergent.

If the limit of $\dfrac{u_{n+1}}{u_n}$ equals 1 some other test must be applied.

If the limit is negative but numerically less than 1, the series is convergent by Theorem IV.

311. Example I. *Consider the series whose n^{th} term is*

$$\frac{p\,(p-1)\ldots(p-n+1)}{\lfloor n}\,x^n.$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left\{\frac{p\,(p-1)\ldots(p-n+1+1)}{\lfloor n+1}\cdot x^{n+1} \div \frac{p\,(p-1)\ldots(p-n+1)}{\lfloor n}\,x^n\right\}$$

$$= \underset{n\to\infty}{\mathrm{L}} \left\{ \frac{p(p-1)\ldots(p-n+1)(p-n)}{p(p-1)\ldots(p-n+1)} \times \frac{\lfloor n}{\lfloor n+1} \times \frac{x^{n+1}}{x^n} \right.$$

$$= \underset{n\to\infty}{\mathrm{L}} \frac{p-n}{n+1} x$$

$$= \underset{n\to\infty}{\mathrm{L}} \frac{\frac{p}{n}-1}{1+\frac{1}{n}} \cdot x$$

$= -x$, since $\frac{p}{n}$ and $\frac{1}{n}$ vanish when n is ∞.

Hence numerically the limit is < 1 if $x < 1$; but > 1 if $x > 1$. That is, the series is convergent or divergent according as x is less than or greater than one.

Example II. *Consider the series*

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \frac{1}{n} + \ldots.$$

$$\underset{n\to\infty}{\mathrm{L}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\mathrm{L}} \left(\frac{1}{n+1} \div \frac{1}{n} \right) = \underset{n\to\infty}{\mathrm{L}} \frac{n}{n+1} = \underset{n\to\infty}{\mathrm{L}} \frac{1}{1+\frac{1}{n}}$$

$$= 1.$$

Therefore this test gives no result.

Now the series

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{16} + \ldots$ to infinity

$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + 8 \times \frac{1}{16} + 16 \times \frac{1}{32} + \ldots,$

$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots.$

$\therefore$ the sum of the series tends to an infinite value.

$\therefore$ the series is divergent.

Notice that the terms of this series continually decrease and become indefinitely small, and yet the series is divergent.

By a similar method it can be proved that

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

is convergent if $p > 1$, but divergent if $p < 1$.

If $p > 1$, then $3^p > 2^p$ and $\frac{1}{3^p} < \frac{1}{2^p}$, &c.;

$\therefore$ Series $< \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots.$

But this series is a G. P. with common ratio $\frac{2}{2^p}$,

and $\frac{2}{2^p} < 1$ if $p > 1$;

$\therefore$ the G. P. is convergent.

Hence the given series is convergent when $p > 1$.

Example III. *Show that the series*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots \text{ is convergent.}$$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \left(\frac{1}{(n+1)(n+2)} \times n(n+1) \right) = 1.$$

We must therefore try some other method.

$$u_n = \frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1};$$

$\therefore$ Series $= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \ldots$,

$$S_n = 1 - \frac{1}{n+1};$$

$\therefore$ $S_\infty = 1$ and series is convergent.

312. Two infinite series are said to be equal when they converge to the same sum.

The Principle of Indeterminate Coefficients may be applied to infinite series.

Suppose the two infinite series

$$a_0 + a_1 x + a_2 x^2 + \ldots,$$

and $$b_0 + b_1 x + b_2 x^2 + \ldots$$

are equal for all values of x for which the series are convergent.

Then

$$(a_0 - b_0) + (a_1 - b_1) x + (a_2 - b_2) x^2 + \ldots = 0$$

for all values of x for which the original series are convergent.

When $x = 0$, the limit of the series is $a_0 - b_0$ (§ 304);

$$\therefore \quad a_0 = b_0.$$

Dividing by x and again putting $x = 0$, we have $a_1 = b_1$, and similarly, $a_2 = b_2$, &c.

Examples XXIX c.

Discuss the convergency or divergency of the following infinite series:

1. $7+3\frac{3}{4}+\frac{1}{2}+\ldots$.

2. $7+3\frac{1}{2}+1\frac{3}{4}+\ldots$.

3. $1+2x+3x^2+4x^3+\ldots$.

4. $1-1+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}\ldots$.

5. $1-3x+\frac{3.4}{1.2}x^2-\frac{3.4.5}{1.2.3}x^3+\frac{3.4.5.6}{1.2.3.4}x^4-\ldots$.

6. $1-6+\frac{6.8}{1.2}-\frac{6.8.10}{1.2.3}+\frac{6.8.10.12}{1.2.3.4}-\ldots$.

7. $1-\frac{3}{2}+\frac{3.4}{2.4}-\frac{3.4.5}{2.4.6}+\frac{3.4.5.6}{2.4.6.8}-\ldots$.

8. $\frac{1}{2.4}-\frac{2}{3.5}+\frac{3}{4.6}-\frac{4}{5.7}+\ldots$.

9. $\frac{1}{2.4}+\frac{2}{3.5}+\frac{3}{4.6}+\frac{4}{5.7}+\ldots$. By H. & K. 5386, [illegible] $\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}$ [illegible] $=\frac{5}{12}$. By 5311, ex. 1, [illegible] of $\frac{1}{n+1}$ [illegible]. Hence [illegible] is divergent.

10. $\frac{5}{6}+\frac{5.7}{6.9}+\frac{5.7.9}{6.9.12}+\ldots$.

11. $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots$.

12. $\frac{1}{1+\sqrt{2}}+\frac{1}{2+\sqrt{3}}+\frac{1}{3+\sqrt{4}}+\ldots$.

13. $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$.

14. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots$.

15. $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$ when $p<1$.

16. $(a+1)^2+(a+2)^2x+(a+3)^2x^2+\ldots$.

17. $\frac{1}{x(x+a)}+\frac{1}{(x+2a)(x+3a)}+\frac{1}{(x+4a)(x+5a)}+\ldots$.

CHAPTER XXX

BINOMIAL THEOREM WITH ANY INDEX. APPLICATIONS

313. Let us consider the series

$$1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r+\ldots.$$

If n is a positive integer, the $(n+2)^{\text{th}}$ term is

$$\frac{n(n-1)\ldots(n-\overline{n+1}+1)}{\lfloor n+1}x^{n+1}.$$

The last factor in the numerator $=0$, hence the $(n+2)^{\text{th}}$ term and all succeeding terms vanish.

Thus when n is a positive integer, the series is finite with $(n+1)$ terms; and it has been shown in Chapter XXVIII that the series is equal to $(1+x)^n$.

If n is not a positive integer, the factor $n-r+1$ can never $=0$; consequently the series extends to infinity.

Before proceeding further with the series we must determine under what conditions this infinite series is convergent.

With the notation of § 247,

$$\frac{u_r}{u_{r-1}}=\frac{n-r-1}{r}\cdot x,$$

$$\lim_{r\to\infty}\frac{u_r}{u_{r-1}}=x\lim_{r\to\infty}\left(\frac{n-1}{r}-1\right)=-x \text{ (provided } n \text{ is finite).}^*$$

Hence if x is numerically less than 1 and n is finite, the series is convergent; but if x is greater than 1, the series is divergent. In the remainder of the chapter, therefore, x is supposed to be less than 1.

* It is interesting to note that if n is infinite and is regarded as a positive integer r cannot be greater than n; but if n is regarded as a fraction then r can be greater than n.

314. To prove that

$$_{m+n}C_r = {}_mC_r + {}_mC_{r-1}\,{}_nC_1 + \dots + {}_mC_{r-s}{}_nC_s + \dots + {}_nC_r.$$

Suppose $(m+n)$ articles divided into two groups, group A containing m articles, and group B containing n. Then r articles may be selected in the following different ways:

r from group A and 0 from group B.
$r-1$ from group A and 1 from group B.
$r-s$ from group A and s from group B.
0 from group A and r from group B.

r from A may be chosen in ${}_mC_r$ ways.
$r-1$ from A, 1 from B may be chosen in ${}_mC_{r-1}\times{}_nC_1$ ways.
$r-s$ from A, s from B may be chosen in ${}_mC_{r-s}\times{}_nC_s$ ways.
r from B may be chosen in ${}_nC_r$ ways.

Hence by addition

$$_{m+n}C_r = {}_mC_r + {}_mC_{r-1}\,{}_nC_1 + \dots + {}_mC_{r-s}\,{}_nC_s + \dots {}_nC_r.$$

Corollary. If we substitute the values for ${}_mC_r$, &c., we have

$$\frac{(m+n)(m+n-1)\dots(m+n-r+1)}{\lfloor r} = \frac{m(m-1)\dots(m-r+1)}{\lfloor r} + \frac{m(m-1)\dots(m-r+2)}{\lfloor r-1}\cdot\frac{n}{1} + \dots + \frac{m(m-1)\dots(m-r+s+1)}{\lfloor r-s}\cdot\frac{n(n-1)\dots(n-s+1)}{\lfloor s} + \dots + \frac{n(n-1)\dots(n-r+1)}{\lfloor r}$$

where r is any positive integer and m and n are positive integers, neither of them less than r.

This is an algebraical equation of the r^{th} degree in m, and it has been proved true for all positive integral values of m not less than r, i. e. for more than r values, therefore it is an identity. Hence it is true for all values of m whether integral or fractional, positive or negative. Similarly, it is shown to be true for all values of n.

This is known as **Vandermonde's Theorem.**

315. Let

$$f(m) \equiv 1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots + \frac{m(m-1)\ldots(m-r+1)}{\lfloor r}x^r + \ldots,$$

and

$$f(n) \equiv 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r + \ldots.$$

Then the coefficient of x^r in the product $f(m) \times f(n)$ is

$$\frac{m(m-1)\ldots(m-r+1)}{\lfloor r} + \ldots + \frac{m(m-1)(m-r-s+1)}{\lfloor r-s} \cdot \frac{n(n-1)\ldots(n-s+1)}{\lfloor s} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}.$$

But this last expression

$$= \frac{(m+n)(m+n-1)\ldots(m+n-r+1)}{\lfloor r}$$

by Vandermonde's Theorem.

$$\therefore \quad f(m) \times f(n) = 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \ldots + \frac{(m+n)(m+n-1)\ldots(m+n-r+1)}{\lfloor r}x^r + \ldots$$

$= f(m+n)$ where m and n may have any values.

Put $m = n = 1$; then $f(1) = 1 + x$,

$f(1) \times f(1) = f(2)$;

i.e. $f(2) = (1+x)^2$,

$f(2) \times f(1) = f(3)$;

i.e. $f(3) = (1+x)^3$;

and similarly, $f(n) = (1+x)^n$ **when n is a positive integer.**

Let n be a fraction $\frac{p}{q}$ where p and q are positive integers,

then $$f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) = f\left(\frac{p}{q} + \frac{p}{q}\right) = f\left(\frac{2p}{q}\right);$$

and similarly, $f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \ldots q$ factors $= f\left(\frac{p}{q} \times q\right) = f(p)$;

$$\therefore \quad \left\{f\left(\frac{p}{q}\right)\right\}^q = (1+x)^p \quad \text{since } p \text{ is a positive integer.}$$

Take the q^{th} root

$$f\left(\frac{p}{q}\right) = (1+x)^{\frac{p}{q}},$$

i.e. $f(n) = (1+x)^n$ **when n is a positive fraction.**

Let $n = -k$ where k is positive,

$$f(n) \times f(k) = f(n+k) = f(0) = 1; \qquad \text{(by substitution)}$$

$$\therefore \quad f(n) = \frac{1}{f(k)} = \frac{1}{(1+x)^k} = (1+x)^{-k},$$

i. e. $f(n) = (1+x)^n$ **when n is negative.**

316. Euler's method of proving that

$$f(m) \times f(n) = f(m+n).$$

When m is a positive integer, $f(m) = (1+x)^m$ (§ 283).

When n is a positive integer, $f(n) = (1+x)^n$.

When $m+n$ is a positive integer, $f(m+n) = (1+x)^{m+n}$.

$\therefore$ When m and n are positive integers,

$$f(m) \times f(n) = f(m+n).$$

But the form of the product of two algebraic expressions has nothing to do with the numerical values that may be given to the letters; the form, therefore, of the product $f(m) \times f(n)$ will be the same whether m and n are positive integers or not. Hence, in all cases, $f(m) \times f(n) = f(m+n)$.

317. In dealing with fractional or negative indices it is usual to put $(a+x)^n$ into the form $a^n\left(1+\frac{x}{a}\right)^n$; by this means the repetition of fractional or negative indices is avoided. Thus

$$(2+3x)^{\frac{1}{2}} \equiv 2^{\frac{1}{2}}(1+\tfrac{3}{2}x)^{\frac{1}{2}}, \quad (4-5x)^{-\frac{3}{2}} = 4^{-\frac{3}{2}}(1-\tfrac{5}{4}x)^{-\frac{3}{2}}.$$

The general term and the greatest term are found in just the same way as with a positive integral index.

318. Example I. *Expand* $(1-x)^{-\frac{1}{2}}$ *to 4 terms.*

$$(1-x)^{-\frac{1}{2}} = 1 - \frac{-\frac{1}{2}}{1}x + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{1.2}x^2 - \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}x^3 + \ldots$$

$$= 1 + \frac{x}{2} + \frac{1(1+2)}{1.2} \cdot \left(\frac{x}{2}\right)^2 + \frac{1(1+2)(1+4)}{1.2.3} \cdot \left(\frac{x}{2}\right)^3 + \ldots$$

$$= 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \ldots.$$

Example II. *Expand* $(3+2x)^{-3}$ *to 5 terms.*

$$(3+2x)^{-3} = \{3(1+\tfrac{2}{3}x)\}^{-3} = 3^{-3}(1+\tfrac{2}{3}x)^{-3}$$

$$= \frac{1}{27}\left\{1 + \frac{-3}{1}\left(\frac{2x}{3}\right) + \frac{-3(-3-1)}{1.2}\cdot\left(\frac{2x}{3}\right)^2 + \frac{-3(-3-1)(-3-2)}{1.2.3}\left(\frac{2x}{3}\right)^3 + \frac{-3(-3-1)(-3-2)(-3-3)}{1.2.3.4}\left(\frac{2x}{3}\right)^4 \ldots\right\}$$

$$= \tfrac{1}{27}\{1 - 2x + \tfrac{8}{3}x^2 - \tfrac{80}{27}x^3 + \tfrac{80}{27}x^4 \ldots\}.$$

Example III. *Find the general term in the expansion of*

$$(1-2x)^{-\frac{3}{2}}.$$

$$(r+1)^{\text{th}} \text{ term} = \frac{(-\frac{3}{2})(-\frac{5}{2}-1)\ldots(-\frac{3}{2}-r+1)}{\lfloor r} \cdot (-2x)^r$$

$$= (-1)^{2r}\frac{3\,(5)\,(7)\ldots(2r+1)}{\lfloor r}\cdot\left(\frac{2x}{2}\right)^r$$

$$= +\frac{3.5.7\ldots(2r+1)}{\lfloor r}\,x^r.$$

Example IV. *Find the greatest term in the expansion of* $(1+x)^{\frac{5}{3}}$ *when* $x = 3$.

$$(r+1)^{\text{th}} \text{ term} = r^{\text{th}} \text{ term} \times \frac{\frac{5}{3}-r+1}{r} \times 3;$$

$\therefore$ terms continue to increase so long as

$$\frac{5-3r}{r} > 1,$$

i.e. $$5 > 4r;$$

$\therefore$ r is not > 1, and the second term is the greatest.

And 2nd term $= \frac{5}{3} \times 3 = 2$.

Examples XXX a.

Expand to 5 terms:

1. $(1-2x)^{\frac{1}{2}}$.
2. $(2-3x)^{-2}$.
3. $\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$.
4. $\left(1+\frac{x}{2}\right)^{-\frac{3}{2}}$.
5. $(3-2x)^{-\frac{1}{3}}$.
6. $(2-x)^{\frac{3}{2}}$.

7. $(1-x^2)^{-3}$.

8. $(1+x)^{-n}$.

9. $(1+x)^{\frac{p}{q}}$.

10. $\sqrt{1-3x}$.

11. $\dfrac{1}{\left(1-\dfrac{x}{3}\right)^3}$.

12. $\{\sqrt[3]{1+2x}\}^2$.

Find the simplest form of the general term (the $(r+1)^{\text{th}}$) in the following expansions:

13. $(1+x)^{-n}$.

14. $(1-2x)^{-2}$.

15. $(1-x)^{\frac{p}{q}}$ when $p<q$.

16. $(1+5x)^{\frac{2}{3}}$.

17. $(2-3x^2)^{-\frac{1}{2}}$.

18. $\left(1-\dfrac{x}{3}\right)^{-3}$.

19. Find the seventh term in the expansion of $(8a^3+3)^{\frac{1}{3}4}$ in descending powers of a.

20. Find the coefficient of x^{20} in the expansion of $(1+x)^{-2}$.

21. Find the square root of 1·1 to 3 places of decimals by means of the Binomial Theorem.

22. Use the Binomial Theorem to find, correct to 3 significant figures, the value of $\dfrac{1\cdot072}{\cdot988}$.

23. Find the magnitude of the greatest term in the expansion of $(1-x)^{-\frac{1}{2}}$ when $x=\frac{12}{13}$.

24. After what term does the expansion of $(1+x)^{\frac{1}{2}}$ begin to converge when $x=\cdot8$?

319. Expansion of multinomials. General Theorem.

When n is a positive integer, the coefficient of $a^pb^qc^r\ldots$ in the expansion of

$(a+b+c+d+\ldots)^n$ is $\dfrac{\lfloor n}{\lfloor p \;\lfloor q \;\lfloor r \ldots}$ where $p+q+r\ldots=n$.

For p a's can be chosen in $\dfrac{\lfloor n}{\lfloor p \;\lfloor n-p}$ ways;

q b's can be chosen from the remaining $n-p$ factors in $\dfrac{\lfloor n-p}{\lfloor q \;\lfloor n-p-q}$ ways;

r c's can be chosen from the remaining $n-p-q$ factors in $\frac{\lfloor n-p-q}{\lfloor r \ \lfloor n-p-q-r}$ ways.

Hence multiplying we have

p a's, q b's, r c's ... can be chosen in $\frac{\lfloor n}{\lfloor p \ \lfloor q \ \lfloor r \ldots}$ ways.

320. Expansion of multinomials. The methods of dealing with multinomials when the index is not a positive integer are best seen from examples.

Example I. *Expand* $(1+x+x^2)^{\frac{1}{3}}$ *as far as* x^3.

$$(1+x+x^2)^{\frac{1}{3}} = (1+\overline{x+x^2})^{\frac{1}{3}}$$

$$= 1 + \frac{x+x^2}{3} + \frac{1(-2)}{1.2}\left(\frac{x+x^2}{3}\right)^2 + \frac{1(-2)(-5)}{1.2.3}\cdot\left(\frac{x+x^2}{3}\right)^3$$

$$= 1 + \frac{x}{3} + \frac{x^2}{3} - \frac{2}{9}x^2 - \frac{x^3}{9} + \frac{5}{81}x^3 + \ldots$$

$$= 1 + \frac{x}{3} + \frac{2x^2}{9} - \frac{13}{81}x^3 \ldots .$$

Example II. *Find the coefficient of* x^n *in the expansion of* $(ab-(a+b)x+x^2)^{-1}$, *when* a *and* b *are both greater than* x.

$$ab-(a+b)x+x^2 = (a-x)(b-x).$$

Let $$\frac{1}{ab-(a+b)x+x^2} \equiv \frac{A}{a-x} + \frac{B}{b-x},$$

then $$1 \equiv A(b-x) + B(a-x).$$

Put $x = a$, $$A = \frac{1}{b-a}.$$

Put $x = b$, $$B = \frac{1}{a-b}.$$

Hence $$\{ab-(a+b)x+x^2\}^{-1} = \frac{1}{(a-b)}\left\{\frac{1}{b-x} - \frac{1}{a-x}\right\},$$

$$\frac{1}{b-x} = \frac{1}{b}\left(1-\frac{x}{b}\right)^{-1} = \frac{1}{b}\left(1+\ldots+\frac{x^n}{b^n}+\ldots\right)$$

and $$\frac{1}{(a-x)} = \frac{1}{a}\left(1-\frac{x}{a}\right)^{-1} = \frac{1}{a}\left(1+\ldots+\frac{x^n}{a^n}+\ldots\right);$$

$$\therefore \text{ coefficient of } x^n = \frac{1}{a-b}\left\{\frac{1}{b^{n+1}} - \frac{1}{a^{n+1}}\right\}$$

$$= \frac{a^{n+1}-b^{n+1}}{a^{n+1}b^{n+1}(a-b)}.$$

Example III. *Find the coefficient of x^{12} in the expansion of $(1-x+x^3-x^5)^{-5}$.*

$$(1-x+x^3-x^5)^{-5} = \left(\frac{1-x^4}{1+x}\right)^{-5} = (1+x)^5(1-x^4)^{-5}.$$

x^{12} is obtained by multiplying 1 by x^{12} or x^4 by x^8.

Hence coefficient $= 5 \times \frac{5.6}{1.2} + 1 \times \frac{5.6.7}{1.2.3} = 110.$

APPLICATIONS OF THE BINOMIAL THEOREM

321. Approximations. The following examples show how the Binomial Theorem may be used to facilitate arithmetical calculations.

Example I. *Find the cube root of 129 correct to the third decimal place.*

$$\sqrt[3]{129} = (125+4)^{\frac{1}{3}}$$

$$= 5(1+\tfrac{4}{125})^{\frac{1}{3}}$$

$$= 5(1+\cdot032)^{\frac{1}{3}}$$

$$= 5\left(1+\tfrac{1}{3}\times\cdot032 + \frac{1.-2}{1.2}\cdot\left(\frac{\cdot032}{3}\right)^2 + \frac{1.-2.-5}{1.2.3}\cdot\left(\frac{\cdot032}{3}\right)^3\ldots\right)$$

$$= 5(1\cdot0106\dot{} - \cdot00011 + \cdot0000018\ldots)$$

$$= 5\times1\cdot01055$$

$$= 5\cdot053 \text{ correct to three decimal places.}$$

Example II. *Find correct to three decimal places the value of* $\dfrac{\sqrt{\cdot941}}{(1\cdot006)^3}$.

$$\begin{aligned}\text{Fraction} &= (1-\cdot059)^{\frac{1}{2}}(1+\cdot006)^{-3}\\ &= (1-\cdot0295-\tfrac{1}{2}\times(\cdot0295)^2\ldots)(1-\cdot018+6\times\cdot000036\ldots)\\ &= (1-\cdot0295-\cdot00043\ldots)(1-\cdot018+\cdot000216\ldots)\\ &= 1-\cdot0295-\cdot00043-\cdot018+\cdot00029+\cdot00023+\cdot000216+\\ &\quad \text{terms beginning with at least four noughts}\\ &= 1\cdot00074-\cdot04793\\ &= \cdot953 \text{ correct to three decimal places.}\end{aligned}$$

The Binomial Theorem is also used for algebraical approximation.

Example III. *If x is so small that its cube and higher powers may be neglected, prove that*

$$\frac{(1-4x)^{\frac{1}{2}}+(1-3x)^{-\frac{1}{3}}}{(1-2x)^{\frac{1}{4}}} = 2+\frac{3x^2}{4},$$

$$(1-4x)^{\frac{1}{2}} = 1-2x+\frac{1.-1}{1.2}4x^2\ldots,$$

$$(1-3x)^{-\frac{1}{3}} = 1+x+\frac{1.4}{1.2}x^2\ldots,$$

$$(1-2x)^{-\frac{1}{4}} = 1+\tfrac{1}{2}x+\frac{1.5}{1.2}\cdot\frac{x^2}{4}\ldots;$$

$$\therefore \text{ fraction} = (2-x)\left(1+\frac{x}{2}+\frac{5x^2}{8}\right).$$

$$= 2+\frac{3x^2}{4} \text{ neglecting cubes and higher powers.}$$

Example IV. *Find the limit of $\dfrac{(x+h)^n-x^n}{h}$ when h is infinitely small.*

$$(x+h)^n = x^n\left(1+\frac{h}{x}\right)^n.$$

We may use the Binomial Theorem since $\dfrac{h}{x}$ is less than unity; and we may neglect the squares and higher powers of $\dfrac{h}{x}$ since h is infinitely small.

$$\text{Hence } \mathop{\mathrm{L}}_{h=0}\frac{(x+h)^n-x^n}{h} = \mathop{\mathrm{L}}_{h=0}\frac{x^n\left(1+\dfrac{nh}{x}\right)-x^n}{h}$$

$$= nx^{n-1}.$$

This is applicable to problems involving COMBINATIONS with REPETITION. See Chrystal's formulation, ch. XXIII, §10.

322. Homogeneous Products. To find the number of homogeneous products of r dimensions that may be made out of the n letters $a, b, c, \ldots$, when each letter may be used any number of times.

$$\frac{1}{1-ax} = 1+ax+a^2x^2+a^3x^3+\ldots,$$

$$\frac{1}{1-bx} = 1+bx+b^2x^2+b^3x^3+\ldots,$$

$$\frac{1}{1-cx} = 1+cx+c^2x^2+c^3x^3+\ldots;$$

$$\therefore \quad \frac{1}{1-ax} \times \frac{1}{1-bx} \times \frac{1}{1-cx} \times \ldots$$

$$= 1+(a+b+c+\ldots)x+(a^2+b^2+c^2+bc+ca+ab+\ldots)x^2+\ldots.$$

It is clear that the coefficient of x^r will be the sum of all the terms of r dimensions that can be obtained from the letters $a, b, c \ldots$, each letter being taken any number of times from 0 to r,

i.e. $$\frac{1}{1-ax} \times \frac{1}{1-bx} \times \frac{1}{1-cx} \times \ldots = 1+S_1x+S_2x^2 \ldots +S_rx^r \ldots$$

where S_r denotes the sum of the homogeneous products of r dimensions.

Let H_r denote the number of these products; or, nH_r

then if $a=b=c=d\ldots$, $S_r = H_r \times a^r$,

i.e. $$\left(\frac{1}{1-ax}\right)^n = 1+H_1ax+H_2a^2x^2 \ldots +H_ra^rx^r+\ldots;$$

but $$(1-ax)^{-n} = 1+nax+\frac{n(n+1)}{1.2}a^2x^2 \ldots$$

$$+\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}a^rx^r+\ldots.$$

Hence $$H_r = \frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}.$$

Note. $$H_r = \frac{\lfloor n+r-1}{\lfloor n-1 \lfloor r} = {}_{n+r-1}C_r.$$

323. Summation of Series. Cf. C. Smith, §288.

Example. *Show that the two infinite series*

$$1+\frac{1}{4}+\frac{1.4}{4.8}+\frac{1.4.7}{4.8.12}+\frac{1.4.7.10}{4.8.12.16}+\dots$$

and $$1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\frac{2.5.8.11}{6.12.18.24}+\dots$$

are equal.

First series

$$=1+\frac{1}{4}+\frac{1(1+3)}{1.2}\cdot\left(\frac{1}{4}\right)^2+\frac{1(1+3)(1+6)}{1.2.3}\cdot\left(\frac{1}{4}\right)^3+\frac{1(1+3)(1+6)(1+9)}{1.2.3.4}\cdot\left(\frac{1}{4}\right)^4+\dots.$$

{This suggests a binomial expansion with fractional index, the denominator being 3. If this is the case, there should be a factor 3 in the denominator for each factor in the numerator. Since the factors in the numerator begin to increase at once, the numerator is -1. Rewrite the series with the object of showing that it is an expansion of the form $(1+x)^{-\frac{1}{3}}$.}

First series

$$=1-\frac{-\frac{1}{3}}{1}\cdot\frac{3}{4}+\frac{(-\frac{1}{3})(-\frac{1}{3}-1)}{1.2}\cdot\left(\frac{3}{4}\right)^2-\frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{1.2.3}\cdot\left(\frac{3}{4}\right)^3\dots$$

$$=(1-\tfrac{3}{4})^{-\frac{1}{3}}=(\tfrac{1}{4})^{-\frac{1}{3}}.$$

Second series

$$=1+2.\frac{1}{6}+\frac{2(2+3)}{1.2}\cdot\left(\frac{1}{6}\right)^2+\frac{2(2+3)(2+6)}{1.2.3}\cdot\left(\frac{1}{6}\right)^3+\dots.$$

$$=1-\frac{-\frac{2}{3}}{1}\cdot\frac{1}{2}+\frac{(-\frac{2}{3})(-\frac{2}{3}-1)}{1.2}\cdot\left(\frac{1}{2}\right)^2-\frac{-\frac{2}{3}(-\frac{2}{3}-1)(-\frac{2}{3}-2)}{1.2.3}\cdot\left(\frac{1}{2}\right)^3\dots$$

$$=(1-\tfrac{1}{2})^{-\frac{2}{3}}=(\tfrac{1}{2})^{-\frac{2}{3}}$$

$$=\{(\tfrac{1}{2})^2\}^{-\frac{1}{3}}=(\tfrac{1}{2})^{-\frac{1}{3}}.$$

Hence the two series are equal.

Examples XXX b.

1. Find the coefficient of x^4 in the expansion of $(1+2x+3x^2)^6$.
2. Find the coefficient of x^{10} in the expansion of $(1-2x^4)^5(1+x)^7$.
3. Find the coefficient of x^{60} in $(1+x+x^2)^{-6}$.
4. Expand $(1+3x+x^2)^{-5}$ as far as the term containing x^3.
5. Find the coefficient of x^5 in $(1+x+x^2)^{\frac{1}{2}}$.
6. Find the coefficient of x^4 in $(1+x+2x^2)^4$.
7. Find the coefficient of x^{10} in the expansion of $(1+x+x^2+x^3)^{-1}$.
8. Find the coefficient of x^{10} in the expansion of $\left(\frac{a+x}{a-x}\right)^3$.
9. Find the coefficient of x^{12} in $(1-x+x^3-x^5)^{-5}$.
10. Find the middle term of $(1+x+x^2)^{12}$.
11. Show, by using the Binomial Theorem, that

$$\sqrt{101} = 10{\cdot}0498756.$$

Find correct to 3 decimal places:

12. $\sqrt{223}$.
13. $\sqrt{22{\cdot}3}$.
14. $\sqrt[3]{340}$.
15. $\sqrt[3]{147}$.
16. $\frac{1{\cdot}003}{{\cdot}997}$.
17. $\left(\frac{1{\cdot}003}{{\cdot}997}\right)^{-3}$.

18. The time of swing of a simple pendulum varies as the square root of the length of the pendulum. If a pendulum 39·37 inches in length swings once in a second, what is the time of swing of a pendulum 40 inches in length?

19. Brass expands ·000019 of its length at 0° C. for each degree that its temperature rises. A brass pendulum beats seconds at 0° C. and its length is then 1 metre. How much will it gain or lose in a day when the temperature is 25° C.?

20. The electrical resistance of a copper wire varies inversely as the area of its cross-section; find the percentage change in the resistance when the temperature rises from 0° C. to 100° C., it being known that the linear dimensions increase ·00168 per cent. for each degree the temperature rises.

21. If x is so small that its square may be neglected, prove that

$$\frac{\sqrt{1+2x}\times\sqrt[4]{16+3x}}{(1-x)^3} = 2+\frac{195}{32}x.$$

22. If x is so small that its cube may be neglected, prove that

$$\frac{(1+x)^{\frac{1}{2}}+(1-x)^{-\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} = 2+x+\frac{3x^2}{4}.$$

23. If N be a large number and a the integer next greater than its fourth root, show that

$$N^{\frac{1}{4}} = a\,\frac{3a^4+5N}{5a^4+3N} \text{ approximately.}$$

24. The roots of $ax^2+bx+c=0$ are $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. Find from the formula what these roots become in the following cases

(i) a infinitely small. (ii) b infinitely small.

(iii) c infinitely small.

25. Find the value of $\dfrac{a^n-b^n}{a-b}$ when $a=b$.

26. Find the number of terms in the expansion of $(a+b+c+\ldots)^n$, there being m terms in the bracket.

27. In a bag there are 25 balls of five different colours and there are five of each colour. In how many ways can 4 balls be chosen when the balls of each colour are (i) distinguishable, (ii) indistinguishable?

28. In the expansion of $(a+b+c+d+\ldots)^n$ where n is less than the number of terms in the bracket, prove that the coefficient of any term in which no letter appears more than once is $\lfloor n$.

29. Show that the number of ways in which a council of $2n+1$ members, representing three parties, may be constituted so that no party has an absolute majority is the coefficient of x^{2n+1} in $(x+x^2+x^3+\ldots+x^n)^3$. Show also that this coefficient is $\frac{1}{2}n(n+1)$.

30. If $(1-x)^{-m} = c_0+c_1x+c_2x^2+c_3x^3+\ldots$ prove that

$$c_0+c_1+c_2+\ldots+c_k = \frac{\lfloor m+k}{\lfloor m \lfloor k},$$

x being less than unity and m and k being positive integers.

31. Find the sum to infinity of the series

$$1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots.$$

32. Show that

$$1+\frac{3}{8}+\frac{3.9}{8.16}+\frac{3.9.15}{8.16.24}+\frac{3.9.15.21}{8.16.24.32}+\ldots=2.$$

33. Show that the sum to infinity of the series

$$\frac{5}{6}+\frac{5.7}{6.9}+\frac{5.7.9}{6.9.12}+\ldots$$

is $3\sqrt{3}-2$.

34. Show that

$$1-2n+3n\,.\,\frac{n+1}{2}-4n\,\frac{(n+1)(n+2)}{2.3}+\ldots \text{ to infinity,}$$

$$=\frac{2-n}{2^{n+1}}.$$

35. Find the value of

$$1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\left(\frac{1}{2}\right)^2-\frac{1.3.5}{2.4.6}\cdot\left(\frac{1}{2}\right)^3+\ldots \text{ to infinity.}$$

36. If p and q be nearly equal and $n>1$, show that

$$\frac{(n+1)(p^2+q^2)+2(n-1)pq}{(n-1)(p^2+q^2)+2(n+1)pq} \text{ is a near approximation to } \left(\frac{p^2+q^2}{2pq}\right)^{\frac{1}{n}}.$$

37. When a balance is used in air of density a and a body of density D is balanced in it by brass weights of density d weighing m grams, the real weight of the body in grams is

$$m\frac{1-\frac{a}{d}}{1-\frac{a}{D}}.$$

Expand this expression to the first power of a and find how much the weight of the body differs from that of the brass weights when $m=1000$, $a=\cdot0012$, $d=8$, $D=\cdot5$.

38. Find the value of the expression

$$\sqrt{x^4+ax^3+bx+c}-\sqrt{x^4+a_1x^3+b_1x+c_1} \text{ when } x=\infty.$$

CHAPTER XXXI

EXPONENTIAL AND LOGARITHMIC SERIES

324. To expand a^x in a series of ascending powers of x.

$$a^x = \{1+(a-1)\}^x$$

$$= 1+x(a-1)+\frac{x(x-1)}{1.2}(a-1)^2+\frac{x(x-1)(x-2)}{1.2.3}(a-1)^3+\ldots$$

$$+\frac{x(x-1)\ldots(x-r+1)}{\lfloor r}(a-1)^r+\ldots. \qquad (A)$$

This series can be arranged in ascending powers of x.

The coefficient of x is

$$(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}\ldots+(-1)^{r-1}\frac{(a-1)^r}{r}+\ldots. \qquad (B)$$

Denote this by c, and the coefficients of x^2, x^3, &c., by c_2, c_3, ...

Then $$a^x = 1+cx+c_2x^2+c_3x^3+\ldots+c_rx^r+\ldots, \qquad (C)$$

where c, c_2, c_3, ... are functions of a, but independent of x.

This is true for all values of x. Replace x by $x+h$.

Then $$a^{x+h} = 1+c(x+h)+c_2(x+h)^2+\ldots+c_r(x+h)^r+\ldots, \qquad (D)$$

but $$a^{x+h} = a^x \times a^h$$

$$= (1+cx+c_2x^2+c_3x^3+\ldots)(1+ch+c_2h^2+c_3h^3+\ldots). \qquad (E)$$

The right-hand sides of D and E are, therefore, equal for all values of x. Equating the coefficients of x, we have

$$c(1+ch+c_2h^2+c_3h^3+\ldots+c_rh^r+\ldots)$$

$$= c+2c_2h+3c_3h^2+\ldots+rc_rh^{r-1}+\ldots.$$

This is true for all values of h. Equating the coefficients of like powers of h on the two sides,

$$2c_2 = c^2; \qquad \therefore\ c_2 = \frac{c^2}{2},$$

$$3c_3 = cc_2; \qquad \therefore\ c_3 = \frac{c^3}{2.3},$$

$$4c_4 = cc_3; \qquad \therefore\ c_4 = \frac{c^4}{2.3.4},$$

$$rc_r = cc_{r-1}; \qquad \therefore\ c_r = \frac{c^r}{2.3\ldots r}.$$

. . . . . .

Substituting these values in (C), we have

$$a^x = 1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \ldots + \frac{c^rx^r}{\lfloor r} + \ldots . \qquad (F)$$

This is true for all values of x. Give x such a value that $cx = 1$; i.e. put $x = \frac{1}{c}$.

Then $$a^{\frac{1}{c}} = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots + \frac{1}{\lfloor r} + \ldots . = \sum_{r=0}^{r=\infty} \frac{1}{\lfloor r}$$

The series on the right hand of this is convergent, since

$$\lim_{r\to\infty} \left\{ \frac{1}{\lfloor r} \div \frac{1}{\lfloor r-1} \right\} = \lim_{r\to\infty} \left(\frac{1}{r}\right)$$

is less than unity.

It has, therefore, a definite value, and that value is always denoted by e.

Hence $a^{\frac{1}{c}} = e$, i.e. $a = e^c$; $\therefore\ c = \log_e a$.

Substituting in (F), we have

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{\lfloor 2} + \frac{(x\log_e a)^3}{\lfloor 3} + \ldots + \frac{(x\log_e a)^r}{\lfloor r} + \ldots .$$

This result is called the **Exponential Theorem or Series.**

If we put $a = e$, we have a useful form of the series:

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots + \frac{x^r}{\lfloor r} + \ldots . = \sum_{r=0}^{r=\infty} \frac{x^r}{\lfloor r}$$

Again, since the coefficient (B) is equal to c, we have

$$\log_e a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} + \ldots + \frac{(-1)^{r+1}(a-1)^r}{r} + \ldots .$$

Put $a = 1 + x$, so that $(a-1) = x$,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1}x^r}{r} - \ldots .$$

This is called the **Logarithmic Series.**

325. Convergency of the Exponential and Logarithmic Series. Series A is a Binomial Expansion, and is convergent for all values of x, only if $(a-1)$ is numerically less than 1, i.e. if a is less than 2.

For the Exponential Series,

$$\mathop{\mathrm{L}}_{r=\infty} \frac{u_r}{u_{r-1}} = \mathop{\mathrm{L}}_{r=\infty} \left\{ \frac{x^r}{\lfloor r} \div \frac{x^{r-1}}{\lfloor r-1} \right\} = \mathop{\mathrm{L}}_{r=\infty} \frac{x}{r},$$

which is less than unity for all values of x.

Hence, as far as the last section shows, the Exponential Expansion is convergent for all values of x, provided a is numerically less than 2. We will now show that the series holds for any value of a.

Let A be a number less than 2 but greater than 1; and let a be a number greater than 2. Then a quantity y can be found such that $A^y = a$, so that $a^x = A^{xy}$.

But since A is less than 2, we may use the Exponential Series for A^{xy}.

$$A^{xy} = 1 + xy \log_e A + \frac{(xy \log_e A)^2}{\lfloor 2} + \ldots + \frac{(xy \log_e A)^r}{\lfloor r} + \ldots;$$

but $A^{xy} = a^x$, and $y \log_e A = \log_e a$. Substituting, we have

$$a^x = 1 + x \log_e a + \frac{(x \log_e a)^2}{\lfloor 2} + \ldots + \frac{(x \log_e a)^r}{\lfloor r} + \ldots,$$

where a is greater than 2.

Hence the Exponential Theorem holds for any value of a, and is convergent for all values of x.

For the Logarithmic Series,

$$\mathop{\mathrm{L}}_{r=\infty} \frac{u_r}{u_{r-1}} = \mathop{\mathrm{L}}_{r=\infty} \left(\frac{x^r}{r} \div \frac{x^{r-1}}{r-1} \right) = x.$$

The series is, therefore, convergent when x is numerically less than 1, but divergent if x is numerically greater than 1.

326. In all theoretical work logarithms are supposed to be referred to base e; and in future we shall suppose that $\log x$ means log to the base e unless some other base is explicitly mentioned. Logarithms to the base e are called **natural** logarithms, for, as we now see, algebraical theory leads us to e as the obvious base to take for logarithms. They are also called Napierian logarithms, after the mathematician Napier, who discovered logarithms early in the seventeenth century.

327. Alternative Proof of the Exponential Theorem.

If the sum of the infinite series

$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots$ is denoted by e, to prove that

$$1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots+\frac{x^r}{\lfloor r}+\ldots=e^x.$$

In the first place both series must be shown to be convergent.

$$\mathop{\mathrm{L}}_{r\to\infty}\left(\frac{1}{\lfloor r}\div\frac{1}{\lfloor r-1}\right)=\mathop{\mathrm{L}}_{r\to\infty}\frac{1}{r}=0,$$

$$\mathop{\mathrm{L}}_{r\to\infty}\left(\frac{x^r}{\lfloor r}\div\frac{x^{r-1}}{\lfloor r-1}\right)=\mathop{\mathrm{L}}_{r\to\infty}\frac{x}{r}=0,$$

whatever x may be.

Hence there is a finite quantity e, since the first series is convergent; and the second series is convergent whatever be the value of x.

Denote $1+m+\frac{m^2}{\lfloor 2}+\frac{m^3}{\lfloor 3}+\ldots$ by $f(m)$,

and $1+n+\frac{n^2}{\lfloor 2}+\frac{n^3}{\lfloor 3}+\ldots$ by $f(n)$.

Then $f(m)\times f(n)=$ an infinite series in which the terms of the r^{th} degree

$$=\frac{m^r}{\lfloor r}+\frac{nm^{r-1}}{\lfloor r-1}+\frac{n^2}{\lfloor 2}\frac{m^{r-2}}{\lfloor r-2}+\ldots+\frac{n^p}{\lfloor p}\frac{m^{r-p}}{\lfloor r-p}+\ldots+\frac{n^r}{\lfloor r}$$

$$=\frac{1}{\lfloor r}\left\{m^r+rm^{r-1}n+\frac{r(r-1)}{1.2}m^{r-2}n^2+\ldots+n^r\right\}$$

$$=\frac{1}{\lfloor r}(m+n)^r,\text{ using the Binomial Theorem.}$$

Hence $f(m)\times f(n)$

$$=1+\frac{m+n}{1}+\frac{(m+n)^2}{\lfloor 2}+\ldots+\frac{(m+n)^r}{\lfloor r}+\ldots$$

$=$ a convergent series

$=f(m+n)$, where m and n are any real quantities.

Similarly, $f(m) \times f(n) \times f(p) \times \ldots = f(m+n+p+\ldots)$.

Put $m = n = p = \ldots = 1$, and suppose there are x factors where x is a positive integer;

then

$$f(1) \times f(1) \times f(1) \times \ldots x \text{ factors} = f(1+1+1+\ldots x \text{ terms}) = f(x);$$

$$\text{i.e. } f(x) = \{f(1)\}^x.$$

Replace $f(x)$ and $f(1)$ by their corresponding series;

$$\text{then } 1+x+\frac{x^2}{\lfloor 2}+\ldots = \left(1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\right)^x = e^x.$$

Hence the theorem is proved when x is a positive integer.

Next suppose $x = \frac{h}{k}$, where h and k are positive integers.

Then

$$f\left(\frac{h}{k}\right) \times f\left(\frac{h}{k}\right) \times \ldots k \text{ factors} = f\left(\frac{h}{k}+\frac{h}{k}+\ldots k \text{ terms}\right);$$

$$\text{i.e. } \left\{f\left(\frac{h}{k}\right)\right\}^k = f(h)$$

$$= e^h, \text{ by first part of proof;}$$

$$\therefore f\left(\frac{h}{k}\right) = \sqrt[k]{e^h} = e^{\frac{h}{k}};$$

$$\text{i.e. } f(x) = e^x,$$

when x is a positive fraction.

Lastly, suppose $x = -y$, where y is positive.

$$f(x) \times f(y) = f(-y+y) = f(0)$$

$$= 1, \text{ by substitution;}$$

$$\therefore f(x) = \frac{1}{f(y)}$$

$$= \frac{1}{e^y}, \text{ since } y \text{ is positive}$$

$$= e^{-y}$$

$$= e^x, \text{ where } x \text{ is negative.}$$

Hence when x is any real quantity

$$1+x+\frac{x^2}{\lfloor 2}+\ldots \text{ to infinity} = e^x.$$

328. To show that

$$\mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \text{ and } \mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{1.2}\left(\frac{1}{n}\right)^2+\ldots.$$

Since n is infinite the investigation of § 313 does not apply, and special investigation as to convergency must be made.

The general term $= \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}\left(\frac{1}{n}\right)^r.$

Hence $$\mathop{\mathrm{L}}_{r\to\infty}\frac{u_{r+1}}{u_r} = \mathop{\mathrm{L}}_{r\to\infty}\frac{n-r+1}{r}\cdot\frac{1}{n} = \mathop{\mathrm{L}}_{r\to\infty}\left(\frac{1}{r}-\frac{1}{n}+\frac{1}{rn}\right) = 0.$$

Therefore the series is convergent.

Again $$u_{r+1} = \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)}{1 \quad 2 \quad \ldots \quad (r-1)r}$$

$$= \left(\frac{1}{1}-\frac{1}{n}\right)\left(\frac{1}{2}-\frac{1}{n}\right)\left(\frac{1}{3}-\frac{1}{n}\right)\ldots\left(\frac{1}{r-1}-\frac{1}{n}\right)\frac{1}{r}.$$

If r is a finite integer greater than 1, $\dfrac{1}{r-1}$ is also finite; when n is infinite $\dfrac{1}{n}$ is infinitely small and may be neglected as compared with any finite quantity.

Hence $\mathop{\mathrm{L}}_{n\to\infty} u_{r+1} = \dfrac{1}{1}\left(\dfrac{1}{2}\right)\left(\dfrac{1}{3}\right)\ldots\left(\dfrac{1}{r-1}\right)\left(\dfrac{1}{r}\right) = \dfrac{1}{\lfloor r}$ when r is finite.

When r is infinite but less than n, $\dfrac{1}{r-1}-\dfrac{1}{n}$ is positive, less than $\dfrac{1}{r-1}$, and infinitely small.

Therefore $\mathop{\mathrm{L}}_{n\to\infty} u_{r+1} < \dfrac{1}{1}\left(\dfrac{1}{2}\right)\left(\dfrac{1}{3}\right)\ldots\left(\dfrac{1}{r}\right)$ when r is infinite but less than n.

$< \dfrac{1}{\lfloor r}$ which is itself infinitely small.

Compare the two series:

$$e = 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots+\frac{1}{\lfloor r}+\ldots.$$

$$\mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots+\mathop{\mathrm{L}}_{n\to\infty} u_{r+1}+\ldots.$$

The series are identical when r is finite and differ only when r is infinite.

Let p be the first value of r that can be regarded as infinite.

Then $e = 1+1+\frac{1}{\lfloor 2}+\ldots+\frac{1}{\lfloor p-1}+\frac{1}{\lfloor p}+\ldots$ to infinity;

but since e is known to be convergent,

$$\frac{1}{\lfloor p}+\frac{1}{\lfloor p-1}+\ldots \text{ must be negligible.}$$

Again $\mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{\lfloor 2}+\ldots+\frac{1}{\lfloor p-1}+R,$

where $$R<\left(\frac{1}{\lfloor p}+\frac{1}{\lfloor p+1}+\ldots\right);$$

$$\therefore\ e-\mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{1}{n}\right)^n < \frac{1}{\lfloor p}+\frac{1}{\lfloor p+1}+\ldots$$

$$< \text{an infinitely small quantity;}$$

$$\therefore \mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

$\mathop{\mathrm{L}}_{n=\infty}\left(1+\frac{x}{n}\right)^n$ can be shown by applying the same processes to equal the series $1+x+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^r}{\lfloor r}+\ldots.$

Hence, assuming the exponential theorem,

$$\mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

Corollary.

By the methods of this section

$$\mathop{\mathrm{L}}_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = 1+x+\frac{x^2}{\lfloor 2}+\ldots;$$

Also $\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^n\right\}^x$, however great n may be.

Hence $1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots=e^x.$

The method of limits, therefore, provides an independent proof of the exponential theorem.

329. Another proof of the Logarithmic Series.

By the Exponential Theorem

$$a^y = 1+\frac{y\log_e a}{1}+\frac{(y\log_e a)^2}{\lfloor 2}+\ldots.$$

Put $a=1+x$, then if x is numerically less than 1 we may use the Binomial Theorem, whatever be the value of y, to expand $(1+x)^y$. Hence

$$1+y.x+\frac{y(y-1)}{1.2}x^2+\ldots=1+y\log_e a+\frac{(y\log_e a)^2}{\lfloor 2}+\ldots.$$

Both series are convergent; we may, therefore, equate coefficients of y.

$$\therefore\quad \log_e a = x-\frac{x^2}{2}+\frac{(-1)(-2)}{\lfloor 3}x^3+\frac{(-1)(-2)(-3)}{\lfloor 4}x^4\ldots;$$

i.e. $$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots(-1)^{n-1}\frac{x^n}{n}\ldots,$$

provided x is numerically less than 1.

330. Calculation of Logarithms.

The logarithmic series can be used only when $x<1$; we can, however, easily deduce a series by means of which any logarithm can be calculated:

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,$$

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots;$$

$$\therefore\quad \log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right).$$

Example. *Calculate* $\log_e 10$.

Find x so that $\frac{1+x}{1-x}=10$; it is found that $x=\frac{9}{11}$,

then $$\log_e 10 = 2\left[\frac{9}{11}+\frac{1}{3}\cdot\left(\frac{9}{11}\right)^3+\frac{1}{5}\left(\frac{9}{11}\right)^5+\ldots\right].$$

The value of $\log_e 10$ can now be worked to any required number of places.

This series converges very slowly. In practice, logarithms are calculated from another series.

If n is greater than 1, $\frac{1}{n}$ is less than 1. Hence

$$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{3}\left(\frac{1}{n}\right)^3 + \ldots,$$

$$\log\left(1-\frac{1}{n}\right) = -\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n}\right)^2 - \frac{1}{3}\left(\frac{1}{n}\right)^3 - \ldots;$$

$$\therefore \quad \log\frac{n+1}{n-1} = 2\left\{\frac{1}{n} + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{1}{5}\frac{1}{n^5} + \ldots\right\}.$$

By putting $n = 2$ we can calculate $\log 3 - \log 2$.

By putting $n = 3$ we can calculate $\log 4 - \log 2$, i.e. $\log 2$.

Since $\log 8 = \log 2^3 = 3\log 2$, $\log 8$ can be calculated.

By putting $n = 9$, we have

$$\log 10 - \log 8 = 2\left\{\frac{1}{9} + \frac{1}{3}\cdot\frac{1}{9^3} + \frac{1}{5}\cdot\frac{1}{9^5}\ldots\right\}.$$

Hence $\log_e 10$ can be calculated.

We can now find the common logarithm of any number, for it has been shown in Part I, § 206, that

$$\log_b N = \frac{\log_a N}{\log_a b}.$$

Hence
$$\log_{10} N = \frac{\log_e N}{\log_e 10}$$
$$= \log_e N \times \frac{1}{\log_e 10}.$$

$\frac{1}{\log_e 10}$ is called the **modulus** of the common system of logarithms. It is often denoted by the letter μ; its value, correct to eight places, is ·43429448. $= \log_{10} e$.

331. Example I. *Find the sum of the series*

$$1 + \frac{x}{\lfloor 2} + \frac{2x^2}{\lfloor 3} + \frac{3x^3}{\lfloor 4} + \ldots.$$

Let
$$S = 1 + \frac{x}{\lfloor 2} + \frac{2x^2}{\lfloor 3} + \frac{3x^3}{\lfloor 4} + \ldots + \frac{(r)\,x^r}{\lfloor r+1} + \ldots,$$

$$u_{r+1} = \frac{r+1-1}{\lfloor r+1}x^r$$

$$= \frac{x^r}{\lfloor r} - \frac{1}{x}\cdot\frac{x^{r+1}}{\lfloor r+1};$$

$$\therefore\ S = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$$
$$- \frac{1}{x}\left(\frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots\right) + \frac{x}{1}$$
$$= e^x - \frac{1}{x}(e^x - 1 - x)$$
$$= e^x\left(1 - \frac{1}{x}\right) + \frac{1}{x} + 1.$$

Example II. *Prove that the coefficient of x^{3n} in the expansion of* $\log(1+x+x^2)^{-\frac{1}{2}}$ *is* $\frac{1}{3n}$.

$$(1+x+x^2) = (1-x^3) \div (1-x);$$
$$\therefore\ \log(1+x+x^2) = \log(1-x^3) - \log(1-x),$$
$$\log(1+x+x^2)^{-\frac{1}{2}} = -\frac{1}{2}\left\{\begin{array}{ll} -x^3 & -\frac{x^6}{2} - \frac{x^{3n}}{n} \\ +x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots & + \frac{x^{3n}}{3n} \end{array}\right\};$$
$$\therefore\ \text{the coefficient of } x^{3n} = -\frac{1}{2}\left(\frac{1}{3n} - \frac{1}{n}\right) = \frac{1}{3n}.$$

Example III. *If* $y = -x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots$, *expand x in a series of ascending powers of y.*

We have $y = e^{-x} - 1$, where x may be any real quantity.

$$\therefore\ -x = \log(1+y)$$
$$= y - \frac{y^2}{2} + \frac{y^3}{3} \ldots, \text{ provided } y < 1;$$
$$\text{i.e.}\quad x = -y + \frac{y^2}{2} - \frac{y^3}{3} \ldots, \text{ provided } y < 1.$$

Examples XXXI.

1. Calculate the value of e to six decimal places.

2. Show that $\frac{1}{e} = .3679$.

3. Show that $\frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \ldots$.

4. Prove that $\frac{e^2-1}{2e} = 1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \ldots$.

5. If $f(x) = \frac{e^x+e^{-x}}{2}$ and $\phi(x) = \frac{e^x-e^{-x}}{2}$,

(i) find a series for $f(x)+\phi(x)$;

(ii) prove that $\{f(x)\}^2 - \{\phi(x)\}^2 = 1$.

6. Assuming that the exponential theorem may be used for imaginary indices, prove that

$$\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} \ldots.$$

What is the equivalent of the series

$$x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} \ldots ?$$

7. Show in any way that

$$\mathop{\mathrm{L}}_{x=\infty} \left(1 + \frac{1}{x}\right)^{nx} = \mathop{\mathrm{L}}_{x=\infty} \left(1 + \frac{n}{x}\right)^{x}.$$

8. Find $\mathop{\mathrm{L}}_{h=0} \frac{e^{x+h} - e^x}{h}$.

9. Find the coefficient of x^6 in the expression

$$\left[1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots\right]^3 + \left[x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots\right]^3.$$

What is the value of the expression ?

10. Find the coefficient of x^r in the expansion of $\frac{a-bx+cx^2}{e^x}$.

11. Expand $\log \frac{1+x}{1-x}$ in a series of ascending powers of x.

12. Show that the logarithmic series is convergent when $x = 1$, but divergent when $x = -1$.

13. If $\log_{10} N = \mu \log_e N$, show that $\mu = \frac{1}{\log_e 10}$, and calculate its value to six decimal places.

14. Calculate the common logarithms of the first ten integers, each to four decimal places.

15. Find the coefficients of x^{4m} and x^{4m+1} in the expansion of $\log\{(1+x)(1+x^2)\}$.

16. Expand $\log(1-x-2x^2)^{\frac{1}{2}}$ when x is numerically less than 1.

17. Find the coefficient of x^n in the expansion of

$$\log\{(1+x)^{1+x}(1-x)^{1-x}\}.$$

18. Prove that, if x is numerically less than unity,

$$\frac{x}{1+x^2}+\frac{1}{3}\left(\frac{x}{1+x^2}\right)^3+\frac{1}{5}\left(\frac{x}{1+x^2}\right)^5+\ldots$$
$$=x-\frac{2}{3}x^3+\frac{x^5}{5}+\frac{x^7}{7}-\frac{2x^9}{9}\ldots.$$

19. Find the limit of $\dfrac{\log(x+h)-\log x}{h}$, when $h=0$.

20. Sum the infinite series whose n^{th} term is

$$\frac{(n+1)(n+2)}{n!}2^n.$$

21. Show that $\log_e 2=\dfrac{1}{1.2}+\dfrac{1}{3.4}+\dfrac{1}{5.6}+\ldots.$

22. Prove that

$$\log(x+1)-\log x=2\left\{\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\ldots\right\}.$$

23. Prove that

$$\frac{e}{2}=1+\frac{2}{\lfloor 3}+\frac{3}{\lfloor 5}+\frac{4}{\lfloor 7}+\frac{5}{\lfloor 9}+\ldots.$$

24. If α, β be the roots of $px^2+qx+r=0$, show that

$$\log(p+qx+rx^2)=\log p-(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2-\frac{\alpha^3+\beta^3}{3}x^3-\ldots.$$

25. In steam engines the mean pressure is $\dfrac{1+\log_e r}{r}$ times the initial pressure, where r is a constant depending on the form of the engine. Calculate to two decimal places the ratio of the mean pressure to the initial pressure when $r=2$.

26. Prove that

$$1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\ldots=5e.$$

27. Find the sum of the infinite series

$$\frac{x}{1.3}+\frac{x^2}{2.4}+\frac{x^3}{3.5}+\ldots$$

28. Sum the series $\frac{3}{\lfloor 2} + \frac{5}{\lfloor 4} + \frac{7}{\lfloor 6} \ldots$ to infinity. $= \sum_1^\infty \frac{2n+1}{\lfloor 2n} = \sum_1^\infty \left(\frac{1}{\lfloor 2n-1} + \frac{1}{\lfloor 2n} \right)$

29. Sum to infinity $\frac{1}{2.4} - \frac{2}{3.5} + \frac{3}{4.6} - \frac{4}{5.7} \ldots$. $= e - 1$.

30. If $a+b+c=0$, prove that

$$\frac{a^5+b^5+c^5}{5} = \frac{a^3+b^3+c^3}{3} \cdot \frac{a^2+b^2+c^2}{2}.$$

CHAPTER XXXII

PARTIAL FRACTIONS AND RECURRING SERIES

Partial Fractions.

332. If an algebraical fraction is expressed as the sum of a number of algebraical fractions whose denominators are the factors of the original denominator, then these fractions are called **Partial Fractions.** Thus, since

$$\frac{2x}{x^2-a^2} = \frac{1}{x+a} + \frac{1}{x-a},$$

$\frac{1}{x+a}$, $\frac{1}{x-a}$ are the Partial Fractions of $\frac{2x}{x^2-a^2}$.

[The process of splitting a fraction into its partial fractions is of great importance in the Integral Calculus, and has, also, useful applications in Algebra.]

If the numerator of the fraction is of higher dimensions than the denominator, it must first be expressed as the sum of an integral expression and a fraction with numerator of lower dimensions than the denominator, e.g.

$$\frac{x^3+a^3}{x^2-a^2} = x + \frac{a^2x+a^3}{x^3-a^2}.$$

We shall, therefore, consider that the numerator is of lower dimensions than the denominator, in which case partial fractions can be found, each having the numerator of lower dimensions than the denominator.

Resolution into partial fractions is effected by Indeterminate Coefficients, by substituting suitable values in an identity or by a combination of the two methods.

Example I. *Resolve* $\dfrac{3x^2-4x+4}{(x-1)(x+1)(x-2)}$ *into partial fractions.*

Assume $\dfrac{3x^2-4x+4}{(x-1)(x+1)(x-2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{x-2}$.

A, B, and C are of lower dimensions than the denominators; and must therefore be constants.

Multiply by $(x-1)(x+1)(x-2)$.

$$3x^2-4x+4 \equiv A(x+1)(x-2)+B(x-1)(x-2)+C(x-1)(x+1).$$

If we equated coefficients, we should have three independent equations for three unknowns; the assumption is therefore justified.

Put $x=1$, $\quad 3-4+4 = A(2)(1-2)$; $\quad \therefore\ A = -\frac{3}{2}$.

Put $x=-1$, $\quad 3+4+4 = B(-2)(-3)$; $\quad \therefore\ B = \frac{11}{6}$.

Put $x=2$, $\quad 12-8+4 = C(1)(3)$; $\quad \therefore\ C = \frac{8}{3}$.

Hence fraction $\equiv -\dfrac{3}{2(x-1)} + \dfrac{11}{6(x+1)} + \dfrac{8}{3(x-2)}$.

Example II. *Resolve* $\dfrac{3x^2-4x+4}{(x-1)^2(x+1)(x-2)}$ *into partial fractions.*

One of the denominators of the partial fractions is $(x-1)^2$; the corresponding numerator would be of the form $Px+Q$.

But $$\frac{Px+Q}{(x-1)^2} = \frac{P(x-1)+P+Q}{(x-1)^2} = \frac{P}{(x-1)^2} + \frac{P+Q}{x-1},$$

where P and Q are constants. Hence we may assume

$$\frac{3x^2-4x+4}{(x-1)^2(x+1)(x-2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2}.$$

Multiply by $(x-1)^2(x+1)(x-2)$.

$$3x^2-4x+4 \equiv A(x+1)(x-2)+B(x-1)(x+1)(x-2) + C(x-1)^2(x-2)+D(x-1)^2(x+1).$$

Put $x=1$, $\quad 3 = A(2)(-1)$; $\quad \therefore\ A = -\frac{3}{2}$.

Put $x=-1$, $\quad 11 = C(-2)^2(-3)$; $\quad \therefore\ C = -\frac{11}{12}$.

Put $x=2$, $\quad 8 = D(1)^2(3)$; $\quad \therefore\ D = \frac{8}{3}$.

Put $x=0$, $\quad 4 = -2A+2B-2C+D$;

i.e. $\quad 2B = 4-3-\frac{11}{6}-\frac{8}{3}$, $\quad \therefore\ B = -\frac{7}{6}$.

Or equate coefficients of x^3,

$$3 = A-2B-4C-D,$$

$$2B = -3-\tfrac{3}{2}+\tfrac{11}{3}-\tfrac{8}{3},$$

$$B = -\tfrac{7}{6}.$$

Hence, fraction

$$\equiv -\frac{3}{2(x-1)^2} - \frac{7}{4(x-1)} - \frac{11}{12(x+1)} + \frac{8}{3(x-2)}.$$

Example III. *Resolve* $\dfrac{3x^2-4x+4}{(x^2+1)(x-2)}$ *into partial fractions.*

Assume $\dfrac{3x^2-4x+4}{(x^2+1)(x-2)} \equiv \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$.

Multiply by $(x^2+1)(x-2)$,

$$3x^2-4x+4 \equiv (Ax+B)(x-2) + C(x^2+1).$$

Put $x=2$, $\quad 8 = 5C$; $\quad \therefore \ C = \frac{8}{5}$.

Equate coefficients of x^2,

$$3 = A + C; \quad \therefore \ A = \tfrac{7}{5}.$$

Equate constants $\quad 4 = -2B + C; \quad \therefore \ B = -\tfrac{6}{5}$.

Hence, fraction $= \dfrac{7x-6}{5(x^2+1)} + \dfrac{8}{5(x-2)}$.

Example IV. *Resolve into partial fractions* $\dfrac{x^2+x+1}{(x-1)^2(x^2-x+2)^2}$.

Suppose we proceeded as follows:

Let

$$\frac{x^2+x+1}{(x-1)^2(x^2-x+2)^2} \equiv \frac{A}{(x^2-x+2)^2} + \frac{B}{x^2-x+2} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)}.$$

Multiply by $(x-1)^2(x^2-x+2)^2$,

$$x^2+x+1 \equiv A(x-1)^2 + B(x-1)^2(x^2-x+2) + C(x^2-x+2)^2 + D(x^2-x+2)^2(x-1).$$

If we equate coefficients, we have six independent equations but only four unknowns, and this would in general lead to inconsistency; therefore our assumption is not justified.

We should notice that some of the denominators are of the second degree and should assume the corresponding numerators to be of the first degree.

Let

$$\frac{x^2+x+1}{(x-1)^2(x^2-x+2)^2} \equiv \frac{Ax+B}{(x^2-x+2)^2} + \frac{Cx+D}{x^2-x+2} + \frac{E}{(x-1)^2} + \frac{F}{x-1},$$

i.e. $x^2+x+1 \equiv (Ax+B)(x-1)^2 + (Cx+D)(x^2-x+2)(x-1)^2 + E(x^2-x+2)^2 + F(x^2-x+2)^2(x-1).$

Now if we equate coefficients, we have six equations and six unknowns, and the assumption is justified.

Equate coefficients of x^5, $\quad 0 = C + F$.

Put $x=1$, $\quad 3=E(2)^2$; $\quad\therefore\quad E=\frac{3}{4}$.

Substitute for E and transpose,

$$-\tfrac{3}{4}\{(x^3-x)^2+4(x^3-x)\}+x^3+x-2$$
$$\equiv(Ax+B)(x-1)^2+(Cx+D)(x^3-x+2)(x-1)^2$$
$$+F(x^3-x+2)^2(x-1).$$

Divide by $x-1$,

$$-\tfrac{3}{4}\{x^3-x^2+4x\}+x+2$$
$$\equiv(Ax+B)(x-1)+(Cx+D)(x^3-x+2)(x-1)$$
$$+F(x^3-x+2)^2.$$

Put $x=1$, $\quad -\frac{3}{4}(4)+3=F(2)^2$; $\quad\therefore\quad F=0$.

$$\therefore\quad C=0.$$

Substitute

$$\frac{-3x^3+3x^2-8x+8}{4}\equiv(Ax+B)(x-1)+(D)(x^3-x+2)(x-1).$$

Divide by $x-1$,

$$\frac{-3x^2-8}{4}\equiv Ax+B+D(x^3-x+2).$$

Equate coefficients $\quad D=-\frac{3}{4}$,

$$A=D=-\tfrac{3}{4},$$
$$B=-2-2D=-\tfrac{1}{2}.$$

$$\therefore\quad \text{Fraction}=-\frac{3x+2}{4(x^2-x+2)^2}-\frac{3}{4(x^3-x+2)}+\frac{3}{4(x-1)^2}.$$

Example V. *Expand* $\dfrac{3x^2-2x+1}{(x-1)(x+3)(x-4)}$ *in ascending powers of* x.

Let $\dfrac{3x^2-2x+1}{(x-1)(x+3)(x-4)}\equiv\dfrac{A}{x-1}+\dfrac{B}{x+3}+\dfrac{C}{x-4}$.

Then $\quad A=\dfrac{2}{-12}$, $\quad B=\dfrac{84}{28}$, $\quad C=\dfrac{41}{21}$.

$$\text{Hence, fraction}=\frac{1}{6}(1-x)^{-1}+\frac{17}{14}(3+x)^{-1}-\frac{41}{21}(4-x)^{-1}$$
$$=\frac{1}{6}(1+x+x^2+\ldots+x^n+\ldots)$$
$$+\frac{17}{42}\left(1-\frac{x}{3}+\frac{x^2}{9}+\ldots+\left(\frac{-x}{3}\right)^n+\ldots\right)$$
$$-\frac{41}{84}\left(1+\frac{x}{4}+\frac{x^2}{16}+\ldots+\left(\frac{x}{4}\right)^n+\ldots\right).$$

$$\text{Hence, fraction}=\sum_{n=0}^{n=\infty}\left\{x^n\left(\frac{1}{6}+(-1)^n\frac{17}{42.3^n}-\frac{41}{84.4^n}\right)\right\}.$$

Example VI. *Find the sum of the first $n+1$ coefficients of the expansion of* $\dfrac{3x-2}{x^2-5x+6}$.

Method I.

Let $\dfrac{3x-2}{x^2-5x-6} = a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots.$

Now $\dfrac{1}{1-x} = 1+x+x^2+\ldots+x^n+\ldots.$ (§264).

If x has such a value that both series are convergent, the product of the two series will equal $\dfrac{3x-2}{x^2-5x-6} \times \dfrac{1}{1-x}$.

The coefficient of x^n in the product of the two series is

$$a_0+a_1+a_2+\ldots+a_n.$$

Hence the required sum of $\overline{n+1}$ coefficients equals the coefficient of x^n in the expansion of $\dfrac{3x-2}{-(1-x)(1-x)(6+x)}$. And this expansion can be carried out as in the previous example.

The artifice of this method is frequently used in summing series; the following method is more straightforward.

Method II.

Resolving into partial fractions,

$$\frac{3x-2}{x^2-5x-6} = \frac{16}{7(x-6)} + \frac{5}{7(x+1)}$$

$$= \frac{5}{7}(1-x+x^2+\ldots)$$

$$-\frac{16}{42}\left(1+\frac{x}{6}+\left(\frac{x}{6}\right)^2+\ldots\right).$$

Sum of $\overline{n+1}$ terms $= \dfrac{5}{7}\dfrac{1-(-x)^{n+1}}{1+x} - \dfrac{16}{7}\dfrac{1-\left(\frac{x}{6}\right)^{n+1}}{6-x}$.

By putting $x=1$ we obtain the sum of the first $\overline{n+1}$ coefficients.

Sum of coefficients $= \dfrac{5}{7}\dfrac{1-(-1)^{n+1}}{2} - \dfrac{16}{7}\dfrac{1-(\frac{1}{6})^{n+1}}{5}$

$$= -\frac{1}{10} + \frac{16}{35}\cdot 6^{-(n+1)} - \frac{5}{14}(-1)^{n+1}.$$

Examples XXXII a.

Resolve into partial fractions:

1. $\dfrac{x+1}{(x+2)(x+3)}$.
2. $\dfrac{3x^3-2x-5}{(x-2)(x+2)(x+3)}$.
3. $\dfrac{3-2x}{x^3-x}$.
4. $\dfrac{2x}{(1+x)^2(1-x)}$.
5. $\dfrac{1+x-x^2}{x^3(1+x)^2}$.
6. $\dfrac{x^3+1}{x^3-1}$.
7. $\dfrac{3x^3-x^2+3}{x(x^2-x+1)^2}$.
8. $\dfrac{8}{(1-x)(1+x)^2(1+x^2)}$.
9. $\dfrac{6a^2}{(x-2a)^2(x-3a)}$.
10. $\dfrac{4}{(x^2+1)^2(x-1)^2}$.

11. By a general method prove that the expression

$$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$$

can be identically found in the form

$$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}.$$

Show that $$A=\frac{pa^2+qa+r}{(a-b)(a-c)}.$$

Give a simple rule for finding A, B, and C.

12. By resolving $\dfrac{x^3}{(x-a)(x-b)(x-c)(x-d)}$ into partial fractions, prove that $\sum \dfrac{a^3}{(a-b)(a-c)(a-d)}=0$.

13. Expand $\dfrac{1+x+2x^2}{(1-2x)(1+x^2)}$ in a series of ascending powers of x.

14. Find the term involving x^n in the expansion of $\dfrac{1+7x}{1-x-2x^2}$.

15. Find the sum of the first n coefficients in the expansion of $\dfrac{1+5x}{(1-3x)(1+2x)}$.

16. Find the coefficient of x^r in the expansion of

$$\frac{1-9x+32x^2}{(1-5x)^2(1+7x)}.$$

RECURRING SERIES

333. If in the series

$$u_1 + u_2 + u_3 + \ldots,$$

a relation such as $u_n = pu_{n-1} + qu_{n-2}$ exists, the series is **a recurring series.**

Thus in the series

$$1 + 4 + 5 + 9 + 14 + 23 \ldots,$$

each term is the sum of the two preceding terms,

$$\text{i.e.} \quad u_n = u_{n-1} + u_{n-2};$$

in the series $\quad 1 + 3x + x^2 - 5x^3 - 7x^4 + 3x^5 \ldots,$

the relation is $u_n = xu_{n-1} - 2x^2 u_{n-2}$.

This is equivalent to

$$u_n - xu_{n-1} + 2x^2 u_{n-2} = 0.$$

The coefficients of this equation taken in their natural order form the **scale of relation.**

Thus the series

$$1 + 3x + x^2 - 5x^3 - 7x^4 + 3x^5 \ldots,$$

is a recurring series with scale of relation $1 - x + 2x^2$.

334. If there are n terms in the scale of relation there will be $n-1$ coefficients in it, besides the first which is always one. If the series is given, it is often required to find the scale of relation. Now n consecutive terms are required before the scale can be used once.

$n+1$ consecutive terms are required before the scale can be used twice.

But to find the $n-1$ unknown coefficients $n-1$ equations are required.

Hence to find a scale of relation with n terms at least $2(n-1)$ consecutive terms of the series must be given.

It is always supposed that only just sufficient terms of the series are given; so that if 8 terms are given, we assume as the scale of relation

$$1 - px - qx^2 - rx^3 - sx^4.$$

335. If a series not containing x such as

$$1+4+5+9+\ldots$$

is given, we find first the scale of relation of

$$1+4x+5x^2+9x^3+\ldots.$$

If this is $1-px-qx^2$; then in the first series $u_n = pu_{n-1}+qu_{n-2}$.

Examples XXXII b.

Write down the first six terms of the recurring series in which

1. The scale of relation is $1-2x+3x^2$, $u_1=1$, $u_2=2x$;

2. The scale of relation is $1-3x-x^2$, $u_1=2$, $u_2=-3x$;

3. The scale of relation is $1-x-2x^2$, $u_1=1$, $u_2=x$; $u_3=3x^2$.

4. The scale of relation is $1-3x+x^2-2x^3$, $u_1=1$, $u_2=2x$, $u_3=3x^2$.

Find the scale of relation of the following series:

5. $1-2x+7x^2-20x^3+\ldots.$

6. $1+2x+x^2-x^3+\ldots.$

7. $2-5x-19x^2-47x^3+\ldots.$

8. $1+2x-3x^2-7x^4+6x^5+\ldots.$

9. Show that any arithmetic progression is a recurring series.

10. Show that any geometric progression is a recurring series.

336. To find the sum of n terms of a recurring series.

Let the series be $a_0+a_1x+a_2x^2+\ldots$;

let the scale of relation be $1-px-qx^2$.

$$S_n = a_0 + a_1x + a_2x^2\ldots + a_{n-1}x^{n-1}$$
$$-pxS_n = \quad -pa_0x-pa_1x^2 \quad -pa_{n-2}x^{n-1}-pa_{n-1}x^n$$
$$-qx^2S_n = \quad -qa_0x^2\ldots -qa_{n-3}x^{n-1}-qa_{n-2}x^n-qa_{n-1}x^{n+1}.$$

When n is greater than 1, $a_n-pa_{n-1}-qa_{n-2}=0$. Hence

$$(1-px-qx^2)S_n = a_0+(a_1-pa_0)x \quad -(pa_{n-1}+qa_{n-2})x^n-qa_{n-1}x^{n+1};$$

$$\therefore\ S_n = \frac{a_0+(a_1-pa_0)x}{1-px-qx^2} - \frac{(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1}}{1-px-qx^2}.$$

It is better to apply the process in all cases than to attempt to use this result as a formula.

If the part of S_n that depends on the value of n becomes indefinitely small when n is infinite there is a sum to infinity; that is, if the recurring series is convergent:—

$$\underset{\infty}{S} = \frac{a_0 + (a_1 - pa_0)x}{1 - px - qx^2}.$$

337. To show that the expansion of $\dfrac{a+bx}{1-px-qx^2}$ in ascending powers of x is a recurring series with scale of relation $1-px-qx^2$.

It is clear that by dividing $a+bx$ by $1-px-qx^2$ an infinite series is obtained.

Let $$\frac{a+bx}{1-px-qx^2} \equiv a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots,$$

then $$a+bx \equiv a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$
$$- pa_0x - pa_1x^2 - \ldots - pa_{n-1}x^n + \ldots$$
$$- qa_0x^2 - \ldots - qa_{n-2}x^n + \ldots.$$

By equating coefficients, it is seen that when $n > 1$

$$a_n - pa_{n-1} - qa_{n-2} = 0,$$

i. e. $$a_nx^n - px \,.\, a_{n-1}x^{n-1} - qx^2 \,.\, a_{n-2}x^{n-2} = 0,$$

i. e. $$u_n - pxu_{n-1} - qx^2u_{n-2} = 0.$$

Hence the expansion is a recurring series with scale of relation

$$1 - px - qx^2.$$

By equating constants, $a = a_0$.

By equating coefficients of x, $b = a_1 - pa_0$.

Hence we see that, whether the recurring series is convergent or not, it can be developed or generated by expanding the fraction $\dfrac{a_0 + (a_1 - pa_0)x}{1-px-qx^2}$; for this reason the fraction is called the **generating function** of the series.

338. If the scale of relation can be resolved into factors, the generating function can be resolved into partial fractions and the general term of the recurring series can be obtained by the method of Example V, § 332.

339. Example. *The scale of relation of the recurring series*

$$1+8x+10x^2+26x^3+46x^4+\ldots,$$

is of the form $1+px+qx^2$. *Find the term of the series involving* x^n; *and the sum of the series to infinity when* x *is numerically less than* $\frac{1}{2}$.

Notice that more terms are given than are necessary.

We have $u_n+pxu_{n-1}+qx^2u_{n-2}=0.$

When $n=2$ $\quad 10+8p+q=0.$ (i)

When $n=3$ $\quad 26+10p+8q=0.$ (ii)

When $n=4$ $\quad 46+26p+10q=0.$ (iii)

From i and ii $\quad 54+54p=0.$

$$\therefore\ p=-1,\quad q=-2.$$

These values also satisfy equation iii.

Let $S \equiv 1+8x+10x^2+26x^3+\ldots$ to infinity,

then $-xS = -x-8x^2-10x^3+\ldots,$

$-2x^2S = -2x^2-16x^3-\ldots.$

$$\therefore\quad S=\frac{1+7x}{1-x-2x^2}$$ for such values of x as make the series convergent.

$$\frac{1+7x}{1-x-2x^2}=\frac{A}{1-2x}+\frac{B}{1+x},$$

where $A = \frac{8}{\frac{8}{3}}=3,\quad B=\frac{-6}{3}=-2.$

Hence the series is the expansion of $\dfrac{3}{1-2x}-\dfrac{2}{1+x}$.

$\therefore$ the term involving $x^n=\{3.2^n-2.(-1)^n\}x^n$.

Also the series is convergent if $2x$ is numerically less than 1, i. e. if x is numerically less than $\frac{1}{2}$.

Examples XXXII c.

Sum to n terms the recurring series:

1. $x+3x^2+5x^3+7x^4+\ldots.$
2. $1+3x+5x^2+7x^3+\ldots.$
3. $1+6x+12x^2+48x^3+120x^4+\ldots.$
4. $2-x+x^2-2x^4\ldots.$
5. $2+5+13+35+97+\ldots.$

6. $1+2+8+44+260+\ldots$.

7. $1+5+7+17+31$.

8. In the recurring series

$$\tfrac{1}{4}-\tfrac{1}{2}x+2x^2+lx^3+5x^4+7x^5+\ldots,$$

the scale of relation is a quadratic expression; determine the unknown coefficient of the fourth term and the scale of relation, and give the general term of the series.

9. Obtain the coefficient of x^{n-1} in the expansion of

$$\frac{1}{1-3x+2x^2}$$

in a series of powers of x.

In the following series, find the sum to infinity, and state for what values of x there is a sum to infinity.

10. $1-3x-3x^2-15x^3\ldots$.

11. $2-x+x^2+12x^3+43x^4+151x^5\ldots$.

12. $-x+x^2-4x^3+7x^4+\ldots$.

13. Resolve $\dfrac{x(x+p)}{(x-a)(x-b)(x-c)(x-d)}$ into partial fractions.

Hence, or otherwise, find the sum to infinity of the series

$$\frac{4.6}{1.3.7.9}+\frac{6.8}{3.5.9.11}+\frac{8.10}{5.7.11.13}+\ldots.$$

14. If the scale of relation of the series $u_1+u_2x+u_3x^2+\ldots$ be $1-3x+2x^2$ find the sum to infinity and show that

$$u_n=u_2(2^{n-1}-1)-2u_1(2^{n-2}-1).$$

CHAPTER XXXIII

THE ELEMENTS OF THE THEORY OF EQUATIONS. HORNER'S METHOD OF SOLUTION. ELIMINATION. DETERMINANTS

Theory of Equations.

340. THROUGHOUT this chapter we suppose that the equation has been simplified so that it is of the form

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0,$$

and we shall denote the left-hand side by $f(x)$.

We also assume the proposition that

If $f(x)$ is a rational integral function of x, there is at least one value α, real or unreal, such that $f(\alpha) = 0$.

341. Every equation of the n^{th} degree has n roots and not more than n roots.

By the last section there is one root, call it α_1.

Since $f(\alpha_1) = 0$, $(x - \alpha_1)$ is a factor of $f(x)$.

Let $f_2(x)$ be the other factor which will be of the $(n-1)^{\text{th}}$ degree.

Hence $$f(x) = 0,$$

if $x = \alpha_1$, or if $f_2(x) = 0$.

But $f_2(x) = 0$ must have a root, call it α_2. Hence

$$f_2(x) = 0,$$

if $x = \alpha_2$, or if $f_3(x) = 0$.

Continuing in this way we find that there are n roots $\alpha_1, \alpha_2 \ldots \alpha_n$, or, to put the same fact in another form, that

$$f(x) \equiv (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).$$

It may happen that r of these factors are the same

i. e. $$f(x) \equiv (x - \alpha)^r \times (n - r) \text{ other factors.}$$

In this case it is usual to say there are r roots equal to α. With this convention we have now proved that every equation of

the n^{th} degree has n roots, which may be real or imaginary, rational or irrational, equal or unequal.

The equation is equivalent to

$$(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)=0.$$

If any quantity k, other than $\alpha_1, \alpha_2 \ldots \alpha_n$, is substituted for x, the product on the left-hand side is not nothing; hence there is no other root than $\alpha_1, \alpha_2 \ldots \alpha_n$, i. e. the equation cannot have more than n roots.

342. Symmetric functions of the roots.

Let $\alpha_1, \alpha_2, \alpha_3 \ldots$ be the n roots.

Then

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n \equiv (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

Equating the coefficients, we have as in § 225.

$$\begin{aligned} \Sigma\alpha &= -p_1, \\ \Sigma(\alpha_1\alpha_2) &= p_2, \\ \Sigma(\alpha_1\alpha_2\alpha_3) &= -p_3, \\ &\text{\&c.} \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_n &= (-1)^n p_n. \end{aligned}$$

These relations are useful for two purposes, (i) to solve an equation when some additional fact about the roots is given, (ii) to find the value of symmetrical functions of the roots.

Example I. *Solve* $2x^3-5x^2-4x+12=0$; *it being given that two of the roots are equal.*

Re-write equation $x^3-\frac{5}{2}x^2-2x+6=0.$

Let α, α, β be the roots.

Then
$$2\alpha+\beta = \tfrac{5}{2}, \quad \text{(i)}$$
$$\alpha^2+2\alpha\beta = -2, \quad \text{(ii)}$$
$$\alpha^2\beta = -6, \quad \text{(iii)}$$

i $\times 2\alpha -$ ii $\quad 3\alpha^2-5\alpha-2=0,$

$$\alpha = 2 \text{ or } -\tfrac{1}{3},$$
$$\beta = -\tfrac{3}{2} \text{ or } \tfrac{19}{6}.$$

Substituting in iii it is seen that the first pair of roots satisfies, but the second does not.

$\therefore$ required solution is $x = 2,\ 2,\ -\frac{3}{2}$,

Example II. *Find the sum of the cubes of the roots of the equation*

$$x^3-2x^2-4x+5=0.$$

We have
$$\alpha+\beta+\gamma=2. \quad \text{(i)}$$
$$\beta\gamma+\gamma\alpha+\alpha\beta=-4. \quad \text{(ii)}$$
$$\alpha\beta\gamma=-5. \quad \text{(iii)}$$
$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta)$$
$$=2\{4-3(-4)\}$$
$$\alpha^3+\beta^3+\gamma^3=-15+32=17.$$

343. General Theorems on the nature of the roots.

Theorem I. If $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are the roots of $f(x)=0$, then $-\alpha_1, -\alpha_2, -\alpha_3, \ldots -\alpha_n$ are the roots of $f(-x)=0$.

Theorem II. If all the coefficients $p_1, p_2, \ldots p_n$ are positive, there cannot be a positive root.

Theorem III. If the terms are alternately positive and negative, there cannot be a negative root.

The student will have little difficulty in supplying the proofs of the above theorems.

Theorem IV. Descartes' Rule of Signs. An equation $f(x)=0$ cannot have more positive roots than there are changes of sign from + to -1 or $-$ to + in the terms of $f(x)$.

No elementary proof of this has been given; but it can easily be verified. Suppose, for instance, we have the equation

$$2x^3-6x^2+7x+2=0.$$

There are two changes of signs.

Multiply by $3x-4$ thus introducing a new positive root.

We then have $6x^4-26x^3+45x^2-22x-8=0$.

There are now three changes of sign.

And it is found in all cases that the effect of introducing a positive root is to increase the number of changes of sign by at least one.

If the product of the factors corresponding to the negative and imaginary roots is multiplied in succession by the factors corresponding to the positive roots, at least one change of sign is introduced by each positive root; therefore the number of positive roots cannot exceed the number of changes of sign.

Corollary. Applying this theorem to the equation $f(-x)=0$, we have 'An equation, $f(x)=0$, cannot have more negative roots than there are changes of sign in the terms of $f(-x)$'.

Example I. *Consider the equation*

$$x^5-6x^4+3x^3+2x^2+3x-1=0.$$

There are 3 changes of sign, and therefore not more than 3 positive roots.

The signs in $f(-x)$ are $+--+--$; and therefore not more than 3 negative roots.

Example II. *Consider the equation*

$$x^6-2x^3+x-5=0.$$

There are 3 changes of sign, and therefore there are not more than 3 positive roots.

In $f(-x)$ there is 1 change of sign, therefore there cannot be more than 1 negative root.

Hence to make up 6 roots, there must be at least 2 imaginary roots.

Theorem V. If $\alpha+\sqrt{\beta}$ is a root of $f(x)=0$, and if $\sqrt{\beta}$ is irrational, then $\alpha-\sqrt{\beta}$ is also a root.

For on substituting $\alpha+\sqrt{\beta}$ for x, $f(x)$ will be of the form $P+Q\sqrt{\beta}$ where P and Q are rational.

Hence if $f(\alpha+\sqrt{\beta})=0$, $P=0$ and $Q=0$.

But the result of substituting $\alpha-\sqrt{\beta}$ for x differs from the previous result only in the sign of $\sqrt{\beta}$.

Hence $f(\alpha-\sqrt{\beta})=P-Q\sqrt{\beta}$

$=0$ since $P=0$ and $Q=0$;

$\therefore$ $\alpha-\sqrt{\beta}$ is a root of $f(x)=0$.

Since β may be positive or negative, we have the following two important statements:

Imaginary roots occur in pairs.

Irrational roots occur in pairs.

It follows that an equation of odd degree must have at least one real rational root.

344. Derived Functions.

Let $f(x) \equiv a_0x^n+a_1x^{n-1}+\ldots+a_rx^{n-r}+\ldots+a_{n-1}x+a_n$.

Substitute $x+h$ for x; and expand the powers of $x+h$ by the Binomial Theorem

$$\begin{aligned} f(x+h) \equiv\ & a_0\left(x^n+nx^{n-1}h+\frac{n(n-1)}{1.2}x^{n-2}h^2+\ldots\right) \\ & +a_1\left(x^{n-1}+(n-1)x^{n-2}h+\frac{(n-1)(n-2)}{1.2}x^{n-3}h^2+\ldots\right) \\ & +a_2\left(x^{n-2}+(n-2)x^{n-3}h+\frac{(n-2)(n-3)}{1.2}x^{n-4}h^2+\ldots\right) \\ & +\ldots \\ & +a_{n-1}(x+h) \\ & +a_n. \end{aligned}$$

The term not involving h is

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n.$$

The coefficient of h is

$$na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\ldots+a_{n-1}.$$

The coefficient of $\frac{h^2}{1.2}$ is

$$n(n-1)a_0x^{n-2}+(n-1)(n-2)a_1x^{n-3}+(n-2)(n-3)a_2x^{n-4}+\ldots.$$

Denoting the coefficients of h, $\frac{h^2}{\lfloor 2}$, ... by $f'(x)$, $f''(x)$, ... $f^{(r)}(x)$, ... we see that

$$f(x+h) \equiv f(x)+f'(x)h+\frac{f''(x)}{\lfloor 2}h^2+\ldots+\frac{f^{(r)}(x)}{\lfloor r}h^r+\ldots+\frac{f^{(n)}x}{\lfloor n}h^n,$$

where $f'(x) = a_0nx^{n-1}+a_1(n-1)x^{n-2}+a_2(n-2)x^{n-3}+\ldots,$

and $f''(x) = a_0n(n-1)x^{n-2}+a_1(n-1)(n-2)x^{n-3}$
$+a_2(n-2)(n-3)x^{n-4}+\ldots.$

$f'(x)$ may be derived from $f(x)$ in the following manner.

Multiply each term of $f(x)$ in order by the index of x in that term, at the same time diminishing the index by unity.

$f''(x)$ can be derived in the same way from $f'(x)$; and it can be shown that $f^{(r)}(x)$ is derived in the same way from $f^{(r-1)}(x)$.

$f'(x)$, $f''(x)$, &c. are called the first, second, &c. derived functions of $f(x)$.

345. Continuity of a rational integral function.

Let $$f(x) \equiv a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n,$$

where a_0, a_1, ...a_n are all real, be the rational integral function.

(i) $f(x)$ is infinite in value only when x is infinite in value, since the function is integral.

(ii) There is one and only one value of $f(x)$ corresponding to any value of x, since the function is rational.

(iii) When x changes slightly in value, $f(x)$ also changes only slightly.

For suppose x to change to $x+h$; then it has been shown in the previous section that

$$f(x+h) = f(x) + \frac{f'(x)}{1}h + \text{higher powers of } h,$$

where the coefficients of the powers of h are all rational integral functions and are therefore not infinite for any finite value of x. Hence h may be taken so small that all the terms after the second may be neglected, § 304, Theorem II.

Hence $f(x+h) - f(x) = f'(x)h$
= an indefinitely small quantity when h is indefinitely small.

346. The graph of $f(x)$.

It follows from the previous section that the graph of

$$y = f(x)$$

is a continuous curve (i.e. a curve without any break), and can be cut by every straight line parallel to the y-axis in one and only one point.

The following statements concerning the roots of an equation may easily be proved by means of the graph of $y = f(x)$.

(i) If $f(a)$ and $f(b)$ are of different signs, then an odd number of real roots are intermediate in value between a and b.

(ii) If $f(a)$ and $f(b)$ are of the same sign, either there are no roots between a and b or there are an even number between a and b.

(iii) If the degree of $f(x)$ is odd there must be at least one real root, since $f(\infty)$ and $f(-\infty)$ are of opposite signs.

347. Transformations of an equation. In solving numerical equations we often wish to find (i) an equation whose roots differ from those of the given equation by a known amount, (ii) an equation whose roots are a known multiple of the roots of the given equation.

If $\alpha, \beta, \gamma, \ldots$ are the roots of $f(x) = 0$, to find an equation whose roots are $\alpha-h, \beta-h, \gamma-h, \ldots$.

Let $y = x-h$, then $x = y+h$. If we substitute for x in the original equation, we shall have the equation required.

Hence one form is

$$a_0(y+h)^n + a_1(y+h)^{n-1} + \ldots + a_{n-1}(y+h) + a_n = 0.$$

On arranging this in descending powers of y, it is of the form

$$A_0 y^n + A_1 y^{n-1} + \ldots + A_{n-1} y + A_n = 0.$$

We wish to determine $A_0, A_1, \ldots A_n$.

Replace y by $(x-h)$, then the left-hand side becomes

$$A_0(x-h)^n + A_1(x-h)^{n-1} + \ldots + A_{n-1}(x-h) + A_n = 0.$$

This must be identical with the original equation,

i.e. $A_0(x-h)^n + A_1(x-h)^{n-1} + \ldots + A_{n-1}(x-h) + A_n \equiv f(x)$.

Hence A_n is the remainder when $f(x)$ is divided by $(x-h)$ and

$$A_0(x-h)^{n-1} + A_1(x-h)^{n-2} \ldots + A_{n-1}$$

is the quotient when $f(x)$ is divided by $(x-h)$.

If this quotient is divided by $x-h$, the remainder is A_{n-1}.

By continual division, therefore, we obtain the successive coefficients $A_n, A_{n-1}, A_{n-2} \ldots$.

The divisions are performed by Horner's Method (Part I, p. 156).

Example I. *Find an equation whose roots are the roots of*

$$3x^3 - 4x^2 + 2x - 1 = 0,$$

each diminished by 4.

1	3	− 4	+ 2	−1
+4		+12	+32	+136
1	3	+ 8	+34	+135
+4		+12	+80	
1	3	+20	+114	
+4		+12		
	3	+32		

Required equation is

$$3y^3 + 32y^2 + 114y + 135 = 0.$$

In practice the divisors are not put down at all, nor is the first coefficient of the dividend repeated.

Since the roots of an equation have nothing to do with the actual letter used as the unknown, it is usual to replace y by x in the final equation.

Example II. *Find an equation whose roots are the roots of*

$$x^4-3x^3+2x-7=0,$$

each diminished by 3.

1	0	−3	2	−7
	3	+9	18	60
3	+6	20	53	
	3	18	72	
6	24	92		
	3	27		
9	51			
12				

Required equation is

$$x^4+12x^3+51x^2+92x+53=0.$$

348. If $\alpha, \beta, \gamma, \ldots$ are the roots of $f(x)=0$, to find an equation whose roots are $k\alpha, k\beta, k\gamma, \ldots$.

Here we have $y=kx,$

and $x^n+p_1x^{n-1}+p_2x^{n-2}\ldots+p_n=0.$

Substituting and multiplying by k^n, we have

$$y^n+kp_1y^{n-1}+k^2p_2y^{n-2}+\ldots+k^np_n=0.$$

$\therefore$ required equation is

$$x^n+kp_1x^{n-1}+k^2p_2x^{n-2}+\ldots+k^np_n=0.$$

This transformation is often used when some of the coefficients of the original equation are fractions.

Example. *Transform the equation*

$$x^4-\frac{3}{5}x^3-\frac{2}{45}x^2+\frac{x}{75}-\frac{4}{675}=0$$

into one with integral coefficients.

We have to find k so that

$$x^4-\frac{3}{5}kx^3-\frac{2}{3^2.5}k^2x^2+\frac{1}{3.5^2}k^3x-\frac{4}{3^3.5^2}k^4=0$$

shall have no fractional coefficients.

Clearly 15 is the smallest suitable value for k and the required equation is

$$x^4-9x^3-10x^2+45x-300=0.$$

349. Horner's Method of solving equations with numerical coefficients.

The steps are as follows:

(i) Find an integer a such that a value of x lies between a and $a+1$. Then $x = a+y$, where y is a decimal fraction.

(ii) Find the equation for y by the method of § 347.

Let h be the digit in the first decimal place of the value of y, so that $y = \cdot h \ldots$.

(iii) Find the equation for z whose roots are ten times the roots of the previous equation, so that $z = h \cdot \ldots$.

(iv) h is then found by treating this last equation exactly as the original equation was treated. By continually repeating the process as many decimal places are obtained as are required.

The following example of the method should be studied closely.

Example. *Find a root, correct to three figures, of*

$$6x^3 - 13x^2 + 26x - 35 = 0.$$

(i) Roots cannot be negative by Theorem III, § 343.

(ii) $f(0) = -35$, $f(1) = -16$, $f(2) = 13$.

$\therefore$ there is a root between 1 and 2 by § 346, i.

(iii) Find the equation whose roots are 1 less than those of the above equation.

$$\begin{array}{rrrr} 6 & -13 & 26 & -35 \\ & +\ 6 & -7 & +19 \\ \hline & -\ 7 & 19 & -16 \\ & +\ 6 & -1 & \\ \hline & -\ 1 & 18 & \\ & +\ 6 & & \\ \hline & 5 & & \end{array}$$

Equation is $6x^3 + 5x^2 + 18x - 16 = 0$.

(iv) Find equation whose roots are ten times those of last equation.

Equation is $6x^3 + 50x^2 + 1800x - 16000 = 0$.

(v) Repeat above processes on this last equation.

$$f(0) = -16000.$$

As a trial root take the greatest integer in $\frac{50}{6}$ viz. 8, and determine by Horner's division whether $f(8)$ is positive or negative.

6	50	1800	−16000	
	48	784	+20672	
	98	2584		i.e. $f(8)$ is positive.

There is a root between 0 and 8. Try 7:

6	50	1800	−16000	
	42	644	+17108	
	92	2444		i.e. $f(7)$ is positive.

There is a root between 0 and 7. Try 6:

6	50	1800	−16000	
	36	516	+13896	
	86	2316	− 2104	i.e. $f(6)$ is negative.
	36	732		
	122	3048		
	36			
	158			

We have now found that the root is 1·6 and that the next digit is the unit digit of the root of

$$6x^3+1580x^2+3048x-2104.$$

The whole work is usually put down as follows:

$$6x^3-18x^2+26x-35=0,$$

$f(0)=-35$, $f(1)=-$, $f(2)=+$

6	−13	26	35	1·66
	− 7	19	19	
	− 1	1800	16000	
	+ 50	2316	13896	
	86	732	2104000	
	122	304800	1886976	
	1580	·9696	·217024	
	·1616	·314496		

Required root is 1·67 correct to 3 significant figures.

Note that the sign of the last column is changed (+35 instead of −35), and consequently the last products, 19, 13896, &c., are subtracted instead of being added. The student may, of course, retain the negative sign and add in the last column as in the others.

Negative roots. If a negative root is found to exist, it is advisable to find the corresponding positive root of the equation $f(-x)=0$.

Cube roots. Any arithmetical cube root (fifth root, seventh root, &c.) may be found by applying Horner's method. Thus, in order to find $\sqrt[3]{91}$, solve the equation $x^3-91=0$.

350. Cardan's Method of solving a cubic equation.

[Lemma. Solve $x^3-1=0$.

Either $x-1=0$ or $x^2+x+1=0$.

$$\therefore\quad x=1, \text{ or } \frac{-1+\sqrt{-3}}{2} \text{ or } \frac{-1-\sqrt{-3}}{2}.$$

Since the equation might be written $x^3=1$, we see that the above roots are the three cube roots of unity.

$\omega=\frac{1}{\omega^2}$. It is easily verified that either imaginary root is the square of the other; hence if one imaginary root is denoted by ω, the other is ω^2.

It follows that the three cube roots of any number a are

$$\sqrt[3]{a},\ \omega\sqrt[3]{a},\ \omega^2\sqrt[3]{a}].$$

(i) Let $ax^3+bx^2+cx+d=0$ be the cubic equation.

Divide by a. $\qquad x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0.$

Put $x=y+k$; then the equation for y is

$$y^3+3ky^2+3k^2y+k^3$$
$$+\frac{b}{a}y^2+\&c. \qquad =0.$$

Choose k so that the coefficient of y^2, viz. $3k+\frac{b}{a}$ shall vanish.

Then the equation is reduced to the form

$$y^3+qy+r=0,$$
$$\text{or}\quad x^3+qx+r=0.$$

(ii) In this last equation put $x=y+z$.

Then $\qquad x^3=y^3+z^3+3yz(y+z)=y^3+z^3+3xyz$

and equation becomes

$$y^3+z^3+(3yz+q)x+r=0.$$

We may impose another restriction on y and z in order to determine them. Choose y and z so that

$$3yz+q=0,$$

then

$$y^3+z^3+r=0.$$

These last may be rewritten,

$$y^3+z^3=-r,$$
$$y^3z^3=-\frac{q^3}{27}.$$

Hence y^3 and z^3 are the roots of

$$k^2+rk-\frac{q^3}{27}=0;$$

$$\therefore\quad y^3=\frac{-r+\sqrt{r^2+\frac{4q^3}{27}}}{2}$$

$$=-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}},$$

$$z^3=-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}.$$

Denote the numerical values of the cube roots by α and β respectively, then $y=\alpha$ or $\omega\alpha$ or $\omega^2\alpha$,

$z=\beta$ or $\omega\beta$ or $\omega^2\beta$;

but yz is rational, $\therefore\ y=\alpha,\quad z=\beta,$

or $y=\omega\alpha,\quad z=\omega^2\beta,$

or $y=\omega^2\alpha,\quad z=\omega\beta.$

Hence the roots of $x^3+qx+r=0$ are

$$\alpha+\beta,\ \omega\alpha+\omega^2\beta,\ \text{or}\ \omega^2\alpha+\omega\beta.$$

Examples XXXIII a.

Use Descartes' rule to determine whether the following equations have imaginary roots:

1. $3x^4-5x^3+6=0.$
2. $4x^3-7x+4=0.$
3. $5x^5-3x^4+5x^3-6x-4=0.$
4. $3x^5-2x^3+7x+8=0.$

5 Prove that in an equation with real coefficients, imaginary roots occur in pairs.

Given that $2+\sqrt{-3}$ is one of the roots of the equation

$$x^4-4x^3+8x+35=0,$$

find all its roots.

6. Solve the equations

$$x^4+x^3-2x^2+3x-1=0,$$

$$x^5-4x^4+2x^3-4x^2+4x-1=0,$$

being given that they have two roots in common.

7. If α, β, γ are the roots of the equation

$$x^3 - 7x^2 + 18x - 7 = 0,$$

prove that $(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = 289$;

and $\alpha^3 + \beta^3 + \gamma^3 = -14$.

8. For what value or values of a has the equation

$$x^4 - 3x^3 + 2x + a = 0$$

a pair of equal roots?

9. If α, β, γ be the roots of the equation $x^3 - px^2 + r = 0$, prove that the equation, of which the roots are $\frac{\beta+\gamma}{\alpha}$, $\frac{\gamma+\alpha}{\beta}$, $\frac{\alpha+\beta}{\gamma}$, is

$$rx^3 + 3rx^2 + (3r - p^3)x + r^3 = 0.$$

10. Form an equation whose roots shall be the products of every two of the roots of the equation

$$x^3 - ax^2 + bx + c = 0.$$

11. Transform the equation $x^3 - px^2 + qx - r = 0$ to one, the roots of which are obtained by subtracting one of the roots of the given equation from the sum of the other two.

Hence, or otherwise, solve the equation

$$32x^3 + 16x^2 - 18x - 9 = 0.$$

12. Show by drawing graphs, or otherwise, that a root of the equation $f'(x) = 0$ lies between every adjacent two of the real roots of $f(x) = 0$.

Hence show that $2x^3 + 3ax^2 + 6a^2x + b = 0$ has only one real root.

13. Solve the equation $2x^5 + 2 + x^4 + x = 12x^3 + 12x^2$.

14. Transform $x^3 - 9x^2 + 4x - 8 = 0$ into an equation wanting the second term.

15. Show that the equation

$$x^4 + 3x^3 - 10x - 5 = 0$$

has two real roots of contrary signs, that it cannot have more real roots, and that these roots lie between -1 and -2.

16. Show that a positive root of an equation cannot be greater than one more than the magnitude of the greatest negative coefficient.

17. Show that if a root is repeated, it is also a root of $f'(x) = 0$.

The roots of the equation $3x^2(x^3+8)+16(x^3-1) = 0$ are not all unequal, find them.

Transform the following equations into equations with integral coefficients:

18. $x^3+2x^2+\frac{1}{2}x+\frac{1}{3} = 0$.

19. $x^3-\frac{7}{8}x^2+\frac{1}{8}\frac{1}{6}x-\frac{3}{7}\frac{1}{2} = 0$.

20. $x^4-\frac{4}{3}x^2-\frac{2}{5}x+\frac{5}{7}\frac{1}{2} = 0$.

21. Form the equation whose roots are each 5 times those of the equation $x^4+2x^3-7x-1 = 0$.

22. Form the equation whose roots are less by 3 than the roots of $x^3-27x-36 = 0$.

23. Transform the following equation so that the second term may be wanting: $x^5+3x^4-7x^3-2x+5 = 0$.

24. Transform $x^3-4x^2+5x-2 = 0$ into an equation with the third term missing.

Find at least one root of each of the following equations:

25. $x^3-9x^2+22x-24 = 0$.

26. $8x^3-26x^2+11x+10 = 0$.

27. $2x^4-13x^3+10x-19 = 0$.

28. $2x^3-3x-6 = 0$.

29. $x^4+12x+7 = 0$.

30. Employ Horner's Method to find correct to two decimal places the negative root of $x^5+x+20 = 0$.

Elimination and Determinants.

351. Suppose the two equations

$$a_1x+b_1 = 0, \qquad \text{(i)}$$

$$a_2x+b_2 = 0 \qquad \text{(ii)}$$

are both satisfied by the same value of x. From the first equation $x = -\frac{b_1}{a_1}$, and from the second $x = -\frac{b_2}{a_2}$. But these values are known to be the same. Hence $-\frac{b_2}{a_2} = -\frac{b_1}{a_1}$;

$$\text{i.e.} \quad a_1b_2-a_2b_1 = 0. \qquad \text{(iii)}$$

Equation iii is the condition that the two original equations can be satisfied by the same value of x.

Whenever two equations are satisfied by the same value of the unknown or variable letter, a third equation can in theory always be found which does not contain that letter. For either equation may be solved and the resulting value of the unknown can then be substituted in the other equation. The unknown is then said to be **eliminated**, and if $R=0$ is the resulting equation, R is called the **eliminant**.

352. The left-hand side of equation iii in the previous section is often written in the following form: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

This is called a **determinant**. It may be regarded as consisting of two rows, viz. $a_1\ b_1$ and $a_2\ b_2$, or of two columns, viz. $\begin{matrix} a_1 \\ a_2 \end{matrix}$ and $\begin{matrix} b_1 \\ b_2 \end{matrix}$, or of four constituents, viz. a_1, a_2, b_1, b_2. A determinant with two rows or columns is said to be of the second order.

From the definition

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1.$$

Similarly

$$\begin{vmatrix} 5 & 3 \\ 6 & 4 \end{vmatrix} \equiv 5\times 4 - 6\times 3 \equiv 2,$$

$$\begin{vmatrix} a+b & c \\ c & a+b \end{vmatrix} \equiv \begin{vmatrix} a & b \\ b & a \end{vmatrix} \equiv a\times a - b\times b = a^2 - b^2.$$

353. Determinants are especially useful for eliminating one or more unknowns and for solving simultaneous equations.

It has been shown in the two previous sections that the result of eliminating x from the two equations

$$a_1x + b_1 = 0,$$
$$a_2x + b_2 = 0$$

is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

The method of writing down the determinant is obvious.

Example. *Find the value of p in order that $px+6=0$ and $6x+p=0$ may be satisfied by the same value of x.*

$$px+6=0,$$

$$6x+p=0.$$

Eliminate x $$\begin{vmatrix} p & 6 \\ 6 & p \end{vmatrix} = 0.$$

Expand the determinant,

$$p^2-36=0;$$

$$\therefore \quad p=\pm 6.$$

354. Eliminate x from the equations

$$a_1x^2+b_1x+c_1=0, \tag{i}$$

$$a_2x^2+b_2x+c_2=0. \tag{ii}$$

[In other words, find the condition that these two equations are satisfied by the same value of x.]

Multiply i by a_2, ii by a_1 and subtract:

$$(a_2b_1-a_1b_2)\,x+(a_2c_1-a_1c_2)=0,$$

$$\text{i.e.} \quad x=-\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}=-\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{iii}$$

Multiply i by c_2, ii by c_1 and subtract:

$$(a_1c_2-a_2c_1)\,x^2+(b_1c_2-b_2c_1)\,x=0.$$

Now unless c_1 and c_2 are both zero, x is not zero; therefore we may divide by x. Hence

$$x=-\frac{(b_1c_2-b_2c_1)}{a_1c_2-a_2c_1}=-\frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}. \tag{iv}$$

These values of x must be equal, hence the required eliminant is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}^2.$$

355. Solve $$a_1x+b_1y+c_1=0,$$

$$a_2x+b_2y+c_2=0.$$

By the usual method it is found that

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_2b_1 - a_1b_2};$$

i.e. $$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1};$$

i.e. $$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

where each denominator is obtained from the preceding one by making the usual cyclic changes.

Example. *Solve* $3x - 4y + 2 = 0,$

$$2x + 3y - 5 = 0.$$

$$\frac{x}{\begin{vmatrix} -4 & 2 \\ 3 & -5 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 3 \\ -5 & 2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix}},$$

$$\frac{x}{20 - 6} = \frac{y}{4 + 15} = \frac{1}{9 + 8}$$

$$\therefore \quad x = \frac{14}{17}, \quad y = \frac{19}{17}.$$

356. Determinant of the third order.

Eliminate x and y from

$$a_1x + b_1y + c_1 = 0, \tag{i}$$

$$a_2x + b_2y + c_2 = 0, \tag{ii}$$

$$a_3x + b_3y + c_3 = 0. \tag{iii}$$

Treating the coefficients of the given equations exactly as in § 353, we obtain a determinant of the third order, and we say that the result of elimination is:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{iv}$$

Again, from ii and iii

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

Substituting in equation i, we have

$$a_1\begin{vmatrix} b_2 & c_2 \\ c_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0. \tag{v}$$

Now $$\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} \equiv -(a_2c_3 - a_3c_2) \equiv -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}. \qquad \text{(vi)}$$

Hence $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

by comparing (iv) and (vi).

Rule for expanding a determinant of the third order. Multiply each constituent of the first row by the determinant obtained by omitting the row and column containing that constituent, the signs of the products being alternately positive and negative.

Example. *Do the graphs of the following equations meet in a point or not?*

$$3x - 4y + 2 = 0,$$
$$2x + 3y - 1 = 0,$$
$$-x + 2y + 3 = 0.$$

If the graphs meet in a point, the equations are all satisfied by the same values of x and y. If that is the case the determinant

$$\begin{vmatrix} 3 & -4 & 2 \\ 2 & 3 & -1 \\ -1 & 2 & 3 \end{vmatrix} \text{ must vanish}$$

Now this determinant

$$= 3\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$
$$= 3 \times 11 + 4 \times 5 + 2 \times 7.$$

This is not zero, therefore the graphs do not meet in a point.

357. If there are n equations of the first degree containing $n-1$ variables; the $n-1$ variables can be eliminated by solving $n-1$ equations and substituting the values so obtained in the n^{th} equation.

The left-hand side of the resulting eliminant can be written down at once as a determinant of the n^{th} order exactly as in the case of three equations. The rule for expanding a determinant of any order is the same as for the third order.

The determinant obtained by leaving out the row and column containing a_r is called the **minor** of a_r and is denoted by A_r.

Hence

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \dots \\ a_2 & b_2 & c_2 & d_2 \dots \\ - & - & - & - \\ a_n & b_n & c_n & d_n \dots \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - \dots .$$

Examples XXXIII b.

By means of determinants solve the following simultaneous equations:

1. $3x - 4y = 18$, $2x + y = 1$.
2. $2x + 3y = 1$, $4x + 9y = 0$.
3. $3x - y = 2a + 4b$, $2x + 3y = 5a - b$.
4. $ax + by + c = 0$, $px + qy + r = 0$.
5. Show that the solution of

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

is

$$\frac{-x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & d_1 & a_1 \\ c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \end{vmatrix}} = \frac{-z}{\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

6. Use determinants to solve

(i) $$\frac{3x - 2y + z}{9} = \frac{3y - 2z + x}{-9} = \frac{3z - 2x + y}{2} = 1;$$

(ii) $$x - y = y - z = z + \frac{x}{2} = 2.$$

7. Show that the result of eliminating x and y from

$$a_1x + b_1y + c_1 = a_2x + b_2y + c_2 = a_3x + b_3y + c_3 = 0,$$

is the same as that obtained by eliminating x and y from

$$a_1x + a_2y + a_3 = b_1x + b_2y + b_3 = c_1x + c_2y + c_3 = 0.$$

What theorem does this prove about determinants of the third order?

8. Prove

(i) $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 + b_1 & b_1 \\ a_2 + b_2 & b_2 \end{vmatrix},$$ (ii) $$\begin{vmatrix} pa_1 & qb_1 \\ pa_2 & qb_2 \end{vmatrix} = pq \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Find the values of the following determinants:

9. $\begin{vmatrix} 4 & 8 & 5 \\ 6 & 4 & 3 \\ 8 & 7 & 4 \end{vmatrix}$.

10. $\begin{vmatrix} 3 & 5 & 7 \\ 5 & 6 & 2 \\ 7 & 2 & 9 \end{vmatrix}$.

11. $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$

12. $\begin{vmatrix} 9 & 10 & 5 \\ 16 & 8 & 4 \\ 15 & 6 & 3 \end{vmatrix}$.

358. The labour involved in finding the value of a determinant (**evaluating** or developing as the process is sometimes called) is considerably lessened by means of some general theorems which are more easily proved if we start with another definition of determinants. For illustration we take a determinant of the fourth order, but exactly the same methods apply to any other order.

Definition $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ is called a determinant and stands for the algebraical sum of all the products of the form $abcd$ which can be obtained by changing the order of the suffixes; the sign to be prefixed to any product being determined by the following two rules:

Rule I. The term $a_1b_2c_3d_4$ is positive.

Rule II. Any interchange of two suffixes, the letters being kept in their alphabetical order, changes the sign.

[For example $a_1b_2c_3d_4$ must have $-$ prefixed; $a_3b_1c_4d_2$ can be obtained thus

$$a_1b_3c_4d_2,\ a_3b_1c_4d_2,\ a_3b_1c_2d_4,$$

and must therefore have the negative sign prefixed.]

The student should now expand $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and convince himself that the same result is obtained as by the previous method of development.

Rule II might be given in the following form.

Any interchange of two letters, the suffixes being kept in their natural order, changes the sign.

For consider the same examples as above

$a_1 b_3 c_2 d_4 = a_1 c_2 b_3 d_4$ and is obtained from $a_1 b_2 c_3 d_4$ by one interchange, hence its sign is $-$.

$a_3 b_1 c_4 d_2 = b_1 d_2 a_3 c_4$, and is obtained from $a_1 b_2 c_3 d_4$ by three interchanges, hence its sign is $-$.

359. Theorem I. If any two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed.

For the interchange of two rows (or columns) produces the same effect as interchanging two suffixes (or letters) in every term of the expanded value of the determinant.

Theorem II. If two rows (or columns) of a determinant are the same, the determinant must equal nothing.

For the interchange of these two rows (or columns) produces no effect, and yet changes the sign. Denoting the determinant by Δ, we have therefore $\Delta = -\Delta$, i.e. $\Delta = 0$.

Theorem III. The value of a determinant is unchanged by writing rows as columns (which involves writing columns as rows at the same time).

For by either of the methods of development (§ 357 or § 358) every term contains one constituent, and only one, from each row and from each column; hence the terms are the same in each of the two determinants; also the two forms for Rule II, § 358, show that the sign of any particular term is the same in each case.

It follows that the determinant may be developed as

$$a_1 A_1 - b_1 B_1 + c_1 C, \text{ \&c.}$$

Theorem IV. If every constituent in any row (or column) be multiplied by the same quantity, the determinant is multiplied by that factor.

Let λ be the factor, then every term such as $a_3 b_1 c_4 d_2$ now becomes $(\lambda a_3) b_1 c_4 d_2$.

Theorem V.

$$\begin{vmatrix} a_1+k_1 & b_1 & c_1 \\ a_2+k_2 & b_2 & c_2 \\ a_3+k_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}.$$

Let A_1, A_2, A_3 stand for the minors of a_1, a_2, a_3 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Left-hand determinant $= (a_1+k_1)A_1-(a_2+k_2)A_2+(a_3+k_3)A_3$

$$= a_1A_1-a_2A_2+a_3A_3+k_1A_1-k_2A_2+k_3A_3$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly $\begin{vmatrix} a_1+k_1 & b_1+l_1 & c_1+m_1 \\ a_2+k_2 & b_2+l_2 & c_2+m_2 \\ a_3+k_3 & b_3+l_3 & c_3+m_3 \end{vmatrix}$ **can be expressed as the sum of eight different determinants.**

Theorem VI. The value of the determinant is unaltered if to each constituent of any row (or column) is added the same multiple of the corresponding constituent of another row (or column). For example:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1+\lambda c_1 & c_1 \\ a_2 & b_2+\lambda c_2 & c_2 \\ a_3 & b_3+\lambda c_3 & c_3 \end{vmatrix}.$$

This follows from Theorems V, IV, III. II.

Example I. *Find the value of* $\begin{vmatrix} 49 & 37 & 62 \\ 39 & 47 & 52 \\ 29 & 57 & 42 \end{vmatrix}$.

$$\Delta = \begin{vmatrix} 49 & 37 & 62 \\ -10 & 10 & -10 \\ -10 & 10 & -10 \end{vmatrix}$$ by subtracting first row from second row and second row from third row

$= 0$, since two rows are identical.

Example II. *Find the value of* $\begin{vmatrix} 1 & 1 & 1 \\ a & a^2 & a^3 \\ b & b^2 & b^3 \end{vmatrix}$.

$$\Delta = ab\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix}$$

$$= ab\begin{vmatrix} 1 & 1 & 1 \\ 0 & a-1 & a^2-1 \\ 0 & b-1 & b^2-1 \end{vmatrix}$$

$$= ab\begin{vmatrix} a-1 & a^2-1 \\ b-1 & b^2-1 \end{vmatrix}$$

$$= ab(a-1)(b-1)\begin{vmatrix} 1 & a+1 \\ 1 & b+1 \end{vmatrix}$$

$$= ab(a-1)(b-1)(b-a).$$

This might be done by applying the factor theorem.

360. Multiplication of determinants. To verify the

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1x_1+b_1y_1+c_1z_1 & a_1x_2+b_1y_2+c_1z_2 & a_1x_3+b_1y_3+c_1z_3 \\ a_2x_1+b_2y_1+c_2z_1 & a_2x_2+b_2y_2+c_2z_2 & a_2x_3+b_2y_3+c_2z_3 \\ a_3x_1+b_3y_1+c_3z_1 & a_3x_2+b_3y_2+c_3z_2 & a_3x_3+b_3y_3+c_3z_3 \end{vmatrix}$$

This last determinant is the sum of 27 determinants ; 15 of these 21 are of the type $\begin{vmatrix} a_1x_1 & a_1x_2 & b_1y_3 \\ a_2x_1 & a_2x_2 & b_2y_3 \\ a_3x_1 & a_3x_2 & b_3y_3 \end{vmatrix}$.

This equals $x_1x_2y_3 \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix}$ which vanishes.

The remaining 6 are of the type

$$\begin{vmatrix} a_1x_1 & b_1y_2 & c_1z_3 \\ a_2x_1 & b_2y_2 & c_2z_3 \\ a_3x_1 & b_3y_2 & c_3z_3 \end{vmatrix}.$$

This equals $x_1y_2z_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. The sum of this and the 5 similar determinants

$$= (x_1y_2z_3 - x_1y_3z_2 - x_2y_1z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

361. The application of determinants to eliminating $n-1$ unknowns from n linear equations has already been explained.

The following example shows how one unknown may be eliminated from two equations and the result expressed as a single determinant.

Example. *Eliminate x from*

$$ax^3 + bx^2 + cx + d = 0, \qquad (i)$$

$$dx^3 + cx^2 + bx + a = 0. \qquad (ii)$$

Multiply i in succession by x^2 and x. Multiply ii by x^2 and x.

$$\begin{aligned} ax^5+bx^4+cx^3+dx^2 &= 0,\\ ax^4+bx^3+cx^2+dx &= 0,\\ ax^3+bx^2+cx+d &= 0,\\ dx^5+cx^4+bx^3+ax^2 &= 0,\\ dx^4+cx^3+bx^2+ax &= 0,\\ dx^3+cx^2+bx+a &= 0. \end{aligned}$$

Treat x, x^2, x^3, x^4, x^5 as different variables. The eliminant is

$$\begin{vmatrix} a & b & c & d & 0 & 0\\ 0 & a & b & c & d & 0\\ 0 & 0 & a & b & c & d\\ d & c & b & a & 0 & 0\\ 0 & d & c & b & a & 0\\ 0 & 0 & d & c & b & a \end{vmatrix} = 0.$$

Examples XXXIII c.

1. Verify the theorems of § 359 in the case of the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Evaluate the following determinants:

2. $\begin{vmatrix} 5 & 9 & -2 & 0\\ 0 & 5 & 9 & -2\\ 0 & -3 & 4 & 20\\ -3 & 4 & 20 & 0 \end{vmatrix}$.

3. $\begin{vmatrix} 1 & a+b & ab & 0\\ 0 & 1 & a+b & ab\\ 1 & b+c & bc & 0\\ 0 & 1 & b+c & bc \end{vmatrix}$.

4. $\begin{vmatrix} 54 & 72 & 37\\ 17 & 37 & 19\\ 19 & 35 & 18 \end{vmatrix}$.

5. $\begin{vmatrix} a & b & c\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3 \end{vmatrix}$.

6. $\begin{vmatrix} -2a & a+b & a+c\\ b+a & -2b & b+c\\ c+a & c+b & -2c \end{vmatrix}$.

7. Show that $\begin{vmatrix} x & y & z\\ x^2 & y^2 & z^2\\ yz & zx & xy \end{vmatrix} \equiv \begin{vmatrix} 1 & 1 & 1\\ x^2 & y^2 & z^2\\ x^3 & y^3 & z^3 \end{vmatrix}$.

8. Without expansion, show that

$$\begin{vmatrix} y+z & b-y-z & c-y-z \\ y+a & -y & y+c \\ z+a & z+b & -z \end{vmatrix} \equiv 2 \begin{vmatrix} a & z & y \\ z & b & -z-y \\ y & -z-y & c \end{vmatrix}.$$

9. Show that

$$ax^2+2hxy+by^2+2gx+2fy+c$$

can be resolved into factors if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

10. Show by means of a determinant that the graphs of

$$3x-4y+2=0,\ 4x+5y+13=0,\ 3x-8y=2,$$

meet in a point.

11. Obtain the determinant of No. 3 by eliminating x from two equations. Deduce its value.

12. Solve by means of determinants

$$\begin{aligned} ax+by+cz &= k, \\ a^2x+b^2y+c^2z &= k^2, \\ a^3x+b^3y+c^3z &= k^3. \end{aligned}$$

13. If the equations $ax^2+bx+c=0$, $dx^2+ex+f=0$ have a root in common, prove that

$$\begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{vmatrix} = 0.$$

14. Evaluate $\begin{vmatrix} 1 & x & 2x^2-1 \\ x & 2x^2-1 & 4x^3-3x \\ 2x^2-1 & 4x^3-3x & 8x^4-8x^2+1 \end{vmatrix}.$

CHAPTER XXXIV

SCALES OF NOTATION. THEORY OF NUMBERS. CONTINUED FRACTIONS

Scales of Notation.

362. The names given to numbers divide numbers into groups of ten, hundred, thousand; and the ordinary Arabic notation represents this name in a very simple manner, much simpler than the Roman notation for example. In dealing with notation care must be taken not to confuse the name of a number with its symbolic representation. Thus the number eleven should convey a definite idea independent of the symbol 11 or XI.

The almost universal custom of dividing numbers into groups of ten, ten times ten, &c., and of naming the numbers accordingly, certainly arose from the habit of counting on the fingers. In theory there is no particular reason why ten should be chosen rather than any other number; and it is instructive to devote a little time to considering other systems of notation.

363. Any positive integer N can be expressed in the form $a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r^1 + a_0$ where r is a positive integer and $a_n, a_{n-1}, \ldots a_0$ are positive integers less than r.

Let r^n be the highest power of r that is less than N,

then $$N = a_n r^n + b_n \text{ where } a_n < r.$$

Then r^{n-1} is the highest power of r that can be less than b_n

$$\therefore \quad b_n = a_{n-1} r^{n-1} + b_{n-1} \text{ where } a_{n-1} < r,$$

and $$N = a_n r^n + a_{n-1} r^{n-1} + b_{n-1}.$$

Continuing the process, we see that

$$N = a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0.$$

Note that any of the quantities $a_0 \ldots a_{n-1}$ may be zero.

The quantity r is called the **radix** of the scale.

The scale is said to be Binary, Ternary, Quaternary, Quinary, Senary, Septenary, Octenary, Nonary, Undenary, or Duodenary, according as the radix is two, three, four, ... eleven or twelve.

No digit can be greater than or equal to the radix, e. g. in the septenary scale, 6 is the highest digit. In the scales higher than the denary the letters *t* and *e* are used as digits to denote the numbers ten and eleven respectively.

In any scale, other than the denary, a number is read out by simply naming the digits in order beginning on the left, e. g. 253 in the septenary scale is read two five three not two hundred and fifty-three.

364. Addition, Subtraction, Multiplication, and Division are performed in any scale exactly as in the denary scale; but care must be taken to divide by the radix of the scale to determine what number to carry and what to put down, just as in ordinary work we divide by 10.

Example I. *Addition in the octenary scale.*

$$\begin{array}{r} 3472 \\ 6305 \\ 3614 \\ 7263 \\ \hline 25076 \end{array}$$

First column. Total is fourteen $= 8\times1+6$; $\therefore$ put down 6 and carry 1.

Second column. Total is fifteen $= 8\times1+7$; $\therefore$ put down 7 and carry 1.

Third column. Total is sixteen $= 8\times2+0$; $\therefore$ put down 0 and carry 2.

Fourth column. Total is twenty-one $= 8\times2+5$ $\therefore$ put down 5 in the fourth column and 2 in the fifth column.

Example II. *Subtraction in the quinary scale.*

$$\begin{array}{r} 4320 \\ 2431 \\ \hline 1334 \end{array}$$

Here five is added to the top digit where in ordinary subtraction we should add ten.

Examples XXXIV a.

1. Express all numbers from ten to twenty in (i) the quinary; (ii) the nonary; (iii) the undenary scale.

2. What number is represented by 320 in (i) the quaternary; (ii) the senary; (iii) the undenary scale?

3. Multiply 7624 by 3465 in the octenary scale; verify the answer.

4. Divide 38480 by nine when the scale is (i) undenary; (ii) nonary.

5. In octenary scale multiply 4467 by 6 and divide the product by 7.

6. Divide the difference between 345678e and e678543 by e in the duodenary scale.

7. Find the cube of 111 in the scale of 2.

8. Find the G. C. M. of 242 and 1111, both numbers being in the scale of 5.

9. Find the square root of 373444 in the nonary scale.

10. Divide $ttet$ 1222 by $teet$ in the duodenary scale.

365. Just as by dividing in the denary scale a vulgar fraction may be expressed as a decimal fraction; so in scale r the vulgar fraction can be expressed as a radix fraction.

Thus in the septenary scale $\frac{2}{7} = \cdot 2$, $\frac{2}{49} = \cdot 21$,

and in the nonary scale $\frac{3}{8} = \cdot 3535\ldots = \cdot\dot{3}\dot{5}$.

This section, taken with § 363, shows that any number may be expressed in the form

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \ldots .$$

366. To reduce a number from one scale to another.

The process is best followed from numerical examples.

Example. *To transfer* 313·253 *from the senary scale to the undenary scale.*

Consider the fractional and integral parts separately.

Let 313 in senary scale $= a_0 + 11a_1 + 11^2 a_2 + \ldots .$

Divide by eleven, recollecting to work in the senary scale.

$$14\frac{7}{11} = \frac{a_0}{11} + a_1 + 11a_2 + \ldots ; \qquad \therefore \quad a_0 = 7.$$

Cancel the fractions and divide by eleven again

$$0+\frac{t}{11}=\frac{a_1}{11}+a_2+\ldots; \qquad \therefore\ a_1=t,\ a_2=0.$$

Hence 313 in senary scale $= t7$ in the undenary.

Let ·253 in senary scale $=\frac{b_1}{11}+\frac{b_2}{11^2}+\ldots.$

Multiply by eleven

$$5\cdot203=b_1+\frac{b_2}{11}+\ldots; \qquad \therefore\ b_1=5.$$

Cancel the integers and again multiply by eleven

$$3\cdot453=b_2+\frac{b_3}{11}+\ldots; \qquad \therefore\ b_2=3.$$

This process can be continued and as many places in the radix fraction can be obtained as may be required.

The work is actually arranged as follows:

11	313		·253
11	14	7	11
	· ·	t	5·203
			11
			3·453
			11
			9·003

Answer $t7\cdot539\ldots.$

It is usual to write the radix, when used as divisor or multiplier, in the denary scale; thus in the above example we write eleven as 11, not as 15 (senary) or 10 (undenary).

Alternative method analogous to reduction in Arithmetic.

Reduce 3 tons 1 cwt. 3 qrs. 2 lb. 5 oz. 3 drs. *to the decimal of a quarter.*

3 tons 1 cwt. 3 qrs.		16	3· drs.
20		16	5·189 oz.
61		28	2·324 lb.
4			·083 qrs.
247			

Answer 247·083.

In an exactly similar way we can reduce 314·253 *from the senary to the undenary scale.*

3 1 3 senary.	6 | 3
6	6 | 5·555
18	6 | 2·*t*0*t*
6	·539
*t*7	

Answer *t*7·539.

In the first method we divide or multiply by the new radix and work in the old scale; in the second method we multiply or divide by the old radix and work in the new scale. Thus if both methods are known, we can work in the denary scale whether we are transforming to or from that scale.

Examples XXXIV b.

1. Express 3125·3125 in the scale of notation whose radix is 8.

2. Transfer 67·75 from the scale of ten to that of five.

3. Multiply the senary number 35·3 by the nonary number 35·3, giving the answer in the denary scale.

4. Convert 384·16 in the nonary scale to the denary scale.

5. In what scale is 147 equivalent to the denary number 124?

6. Reduce $\cdot 2\dot{5}\dot{3}\dot{4}$ in the senary scale to a vulgar fraction in the denary scale.

7. If the double of the denary number 145 is expressed in a certain scale it is represented by the same digits 145; what is the radix of the scale?

8. A dealer has a series of weights 1, 2, 4, 8, ... pounds up to 128, but has only one of each kind. Show that he can weigh any exact number of pounds less than 255 lb. What weights must he use to weigh 159 pounds?

9. If weights may be placed in either scale pan, show how any number of pounds up to 121 lb. may be weighed with the weights 1, 3, 9, 27, 81 lb. How can 58 lb. be weighed?

10. In any scale of notation whose radix is (r) prove that if the sum of the digits of any whole number be divisible by $(r-1)$ the number itself will be divisible by $r-1$.

11. The first and second of the three digits by which a perfect square is expressed are 1 and $2n$ respectively; find the third digit. And show that, whatever be the radix, a perfect square would be obtained by reversing the order of these digits.

12. Prove the following rules for finding the factors of an integral number N expressed in the scale of r:

(a) Any factor of r will also be a factor of N if it exactly divides the extreme right-hand digit.

(b) Any factor of r^2 will also be a factor of N if it exactly divides the number denoted by the two extreme right-hand digits.

(c) N is divisible by $r-1$ if the sum of the digits is divisible by $r-1$.

(d) N is divisible by $r+1$ if the difference between the sums of the odd and even digits is exactly divisible by $r+1$.

13. If n be the sum of the digits of a number N, show that $N-n$ is divisible by the radix $r-1$.

14. Show that a vulgar fraction will reduce to a terminating radix fraction provided the denominator contains no factor which is not a factor of the radix.

15. If the radix is equal to xyz, and if the denominator of a vulgar fraction is $x^a y^b z^c w$ where w does not contain as a factor x, y, or z; show (i) that there will be n non-recurring figures in the equivalent radix fraction where n is the greatest of the three quantities a, b, c; (ii) that the number of recurring figures cannot exceed $w-1$ and must be a factor of $w-1$.

THEORY OF NUMBERS

In dealing with the properties of numbers, we suppose that all the numbers are positive integers.

367. If two numbers have no common factor except unity, they are said to be prime to one another.

If a number has no factors other than itself and unity, it is a **prime number** or **prime**. A number which has factors is a **composite** number.

If a number N can be exactly divided by any other number n, it is a multiple of n; this is denoted as follows

$$N = M(n).$$

If N is not a multiple of n, it can be expressed in the form $N \equiv qn + r$ where q is the integral quotient when N is divided by n, and r is the remainder less than n. When this form of relation is used n is called the **modulus**. For example, with modulus 5 all numbers can be put in one or other of the forms $5q$, $5q+1$, $5q+2$, $5q+3$, $5q+4$, or more simply $5q-2$, $5q-1$, $5q$, $5q+1$, $5q+2$.

368. It is useful practice to prove the following simple theorems.

(i) If $a = nb + r$, and if a is prime to b, then r is also prime to b.

(ii) If a is prime to b and to c, it is prime to bc.

(iii) If a is prime to b but is a factor of bc, then a must be a factor of c.

(iv) If a is prime to bcd ..., it must be prime to each of the factors $b, c, d, \ldots$.

(v) If a number N is divisible by each of the numbers $a, b, c, d, \ldots$ and if $a, b, c, d, \ldots$ are prime to one another then N is divisible by $abcd$.

(vi) If a and b are prime to each other, a^m and b^n are also prime to each other, m and n being any positive integers.

(vii) If a is prime to b, then the fraction $\frac{a}{b}$ is in its lowest terms.

For if possible, suppose $\frac{a}{b} = \frac{a'}{b'}$, then $a' = \frac{ab'}{b}$.

Therefore b divides ab'; but b is prime to a therefore b divides b'. Hence b' cannot be less than b, and $\frac{a'}{b'}$ is not in lower terms than $\frac{a}{b}$.

Example. *If n be even, show that $n(n^2+20)$ is divisible by 48.*

Because n is even, $n = 2m$ where m is any integer;

$$\therefore \quad n(n^2+20) = 2m(4m^2+20) = 8m(m^2+5).$$

If m is even, m is divisible by 2; if m is odd, m^2 is also odd and m^2+5 is even; hence in any case $m(m^2+5)$ is divisible by 2.

Hence $n(n^2+20) = 8M(2) = M(16)$.

It remains to show that $m(m^2+5)$ is divisible by 3.

If $m = 3q$, m is divisible by 3.

If $m = 3q \pm 1$, $m^2+5 = 9q^2 \pm 6q + 6$ which is divisible by 3.

Hence in all cases, $m(m^2+5)$ is divisible by 3.

Therefore $n(n^2+20)$ is divisible by 16×3, i.e. by 48.

369. (i) The number of primes is infinite.

For suppose p to be the greatest prime number.

Then the product of all the prime numbers up to p is divisible by all of them.

Therefore $1+1.2.3.5.7.11.... p$ is prime to all of them.

This number is therefore either itself prime or divisible by a prime greater than p.

In either case p is not the greatest prime number.

Therefore there is no limit to the number of primes.

(ii) No rational integral function of x can be found which will be prime for all values of x.

If possible, let $a+bx+cx^2+dx^3...$ be such a function.

Suppose p is the value when $x = m$.

Then $p = a+bm+cm^2+....$

Let q be the value when $x = m+p$.

Then
$$\begin{aligned} q &= a+b(m+p)+c(m+p)^2+... \\ &= a+bm+cm^2+... \\ &\quad +\text{terms containing } p \\ &= p+M(p) \\ &= M(p); \end{aligned}$$

i.e. q is not prime.

Therefore the function does not represent primes for all values of x.

370. (i) A composite number can be resolved into prime factors in only one way.

If possible let $N = abcd...$ and also $= \alpha\beta\gamma\delta...$,

where $a, b, c, d, ...$ are primes, some of which may be equal;

and $\alpha\beta\gamma\delta...$ are primes, some of which may be equal.

Now α divides N, i.e. α divides $abcd...$,

$\therefore$ α must divide one of the factors.

Suppose a is a factor that α divides,

but a is prime and can be divided only by itself or unity.

$\therefore \quad a = \alpha.$

Similarly b, c, d, ... can be shown to be respectively equal to β, γ, δ ....

(ii) To find the number of divisors of any given number.

Suppose the given number $= a^p b^q c^r \ldots$ where a, b, c are primes.

Then all factors will be of the same form with indices not greater than p, q, r respectively. Hence all the possible factors are the different terms of the product

$$(1+a+a^2 \ldots +a^p)\,(1+b+b^2+\ldots+b^q)\,(1+c+c^2 \ldots +c^r)\ldots.$$

The number of these terms is $(p+1)(q+1)(r+1)\ldots$.

This includes the two terms 1 and $a^p b^q c^r \ldots$.

Hence total number of divisors including unity and N is

$$(p+1)(q+1)(r+1)\ldots.$$

(iii) It follows from (ii) that the sum of all the divisors of $a^p b^q c^r \ldots$ is

$$\frac{(a^{p+1}-1)(b^{q+1}-1)(c^{r+1}-1)\ldots}{(a-1)(b-1)(c-1)\ldots}.$$

371. (i) The product of r consecutive integers is divisible by $\lfloor r$.

The number of combinations of n things taken r at a time is an integer.

$$\therefore \quad \frac{n(n-1)\ldots(n-r+1)}{\lfloor r} = \text{an integer.}$$

Since n and r may have any integral values provided $r \not> n$ this proves the theorem.

This theorem may easily be proved by induction.

(ii) If p is a prime number $(a+b)^p - a^p - b^p$ is divisible by p.

$$\text{For} \quad (a+b)^p - a^p - b^p = \sum_{r=1}^{r=p-1} \left\{ \frac{p(p-1)\ldots(p-r+1)}{\lfloor r} a^{p-r} b^r \right\}. \qquad \text{(A)}$$

Now $\dfrac{p(p-1)\ldots(p-r+1)}{\lfloor r}$ is an integer by (i).

And since p is prime, no factor of $\lfloor r$ divides it; also p is greater than r and cannot therefore divide any factor of $\lfloor r$. Hence the denominator $\lfloor r$ is completely cancelled without using the factor p, i.e. each term in (A) is divisible by p.

(iii) If p is a prime number,

$$(a+b+c+\ldots)^p - a^p - b^p - c^p - \ldots = M(p).$$

For $[a+(b+c+d\ldots)]^p - a^p - (b+c+d+\ldots)^p = M(p)$ by (ii);

i.e. $(a+b+c+d\ldots)^p - a^p - [b^p+(c+d+\ldots)^p + M(p)] = M(p)$.

i.e. $(a+b+c+d\ldots)^p - a^p - b^p - (c+d\ldots)^p = M(p)$.

Continuing this process we have the required result.

(iv) If p is prime, $N^p - N = M(p)$.

In the result of (iii), put $a = b = c \ldots = 1$, and suppose there are N terms, then $a+b+c+\ldots = N$ and $a^p+b^p+c^p\ldots = N$.

Hence $$N^p - N = M(p).$$

(v) If p is a prime number and N is prime to p then

$$N^{p-1} - 1 = M(p) \qquad \text{(Fermat's Theorem.)}$$

For $N^p - N = M(p)$ whether N is prime to p or not;

i.e. $N(N^{p-1}-1) = M(p)$ whether N is prime to p or not,

but N is prime to p.

$\therefore$ $N^{p-1} - 1 = M(p)$ when N is prime to p.

372. (i) If a is prime to b, the remainders obtained when $a, 2a, 3a, \ldots (b-1)a$, are respectively divided by b, are all different.

For suppose that the remainders obtained when pa and qa are divided by b are the same, viz. r.

Then $pa = bx + r$,

$qa = by + r$.

$\therefore$ $(p-q)a = (x-y)b$.

Hence b is a factor of $(p-q)a$.

But p and q are less than b by the conditions of the question, therefore b cannot be a factor of $p-q$.

Hence b must be a factor of a which is contrary to the conditions of the question.

Thus the remainders are all different.

(ii) If p be a prime number, $1 + \lfloor \frac{p-1}{} \rfloor$ is divisible by p.

(Wilson's Theorem.)

Let a be any one of the numbers

$$1, 2, 3, 4, \ldots (p-1).$$

Then a is prime to p; therefore by (i) the remainders obtained when $a, 2a, 3a, 4a, \ldots (p-1)a$ are divided by p, are all different.

Hence one and only one of the products $a, 2a, \ldots (p-1)a$ gives remainder one when divided by p. Let it be ma.

Then $ma = M(p)+1$ where $m \not> (p-1)$.

If $m = a$, $a^2 = M(p)+1$, therefore $(a+1)(a-1)$ is divisible by p, which is the case only when $a = 1$ or $p-1$.

Hence if a is neither 1 nor $p-1$, m and a are different.

It follows that for any one of the numbers 2, 3, 4, ... $p-2$, another can be chosen such that their product is of the form $M(p)+1$.

Hence the $(p-3)$ integers 2, 3, 4, ... $p-2$, can be grouped in pairs so that the product of each pair is of the form $M(p)+1$.

Multiplying all these pairs together

$$2.3.4 \ldots (p-2) = M(p)+1;$$

$$\therefore \quad 1.2.3.4 \ldots (p-2)(p-1) = \{M(p)+1\} \times (p-1),$$

$$= M(p)+p-1,$$

$$\text{i.e.} \quad 1 + \lfloor \underline{p-1} = M(p).$$

373. Example. *To find the highest power of* 5 *which is contained in* $\lfloor \underline{144}$.

Of the first 144 integers;

5, 10, 15, ... 140 contain 5 once as a factor.

25, 50, 75, 100, 125 contain 5 twice as a factor.

125 contains 5 thrice as a factor.

The number of integers in the first of these lines is $\frac{140}{5} = 28$, which is the greatest integer in $\frac{144}{5}$.

The number of integers in the second of these lines is $\frac{125}{25} = 5$, which is the greatest integer in $\frac{144}{5^2}$.

The number of integers in the third of these lines is $\frac{125}{125} = 1$, which is the greatest integer in $\frac{144}{5^3}$.

Now when the product $\lfloor \underline{144}$ is taken, 5 occurs once for each number in the first line; an additional time for each number in the second line; and an additional time for each number in the third line.

Hence the highest powers of 5 in $\lfloor \underline{144} = 28+5+1 = 34$.

374. (i) If $N = a^n$ and a is a prime number, the number of numbers less than N and prime to it is $N\left(1-\frac{1}{a}\right)$.

The numbers from 1 to N are a^x in number.

Of these $a, 2a, 3a, \ldots a^{x-1}a$, are the only numbers not prime to N, and there are a^{x-1} of them.

Hence the number of numbers less than N and prime to it is

$$a^x - a^{x-1} = a^x\left(1-\frac{1}{a}\right) = N\left(1-\frac{1}{a}\right).$$

(ii) If $N = a^x b^y$ and a and b are primes, the number of numbers less than N and prime to it is $N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)$.

The multiples of a are $a \,.\, 2a \,.\, 3a \ldots a^{x-1}b^y a$, and there are $a^{x-1}b^y$ of them.

Similarly, there are $a^x b^{y-1}$ multiples of b.

Some numbers, namely those which are multiples of ab, are included in the multiples both of a and of b.

The multiples of ab are $ab, 2ab, 3ab, \ldots a^{x-1}b^{y-1}ab$, so that there are $a^{x-1}b^{y-1}$ of them.

Hence there are $a^{x-1}b^y + a^x b^{y-1} - a^{x-1}b^{y-1}$ numbers in the first N numbers which are not prime to N.

Therefore the numbers which are prime

$$\begin{aligned} &= a^x b^y - a^{x-1}b^y - a^x b^{y-1} + a^{x-1}b^{y-1} \\ &= a^x b^y\left(1-\frac{1}{a}-\frac{1}{b}+\frac{1}{ab}\right) \\ &= N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right). \end{aligned}$$

(iii) If $N = a^x b^y c^z \ldots$ and $a_1 b_1 c_1 \ldots$ are primes, the number of numbers less than N and prime to it is

$$N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots.$$

This follows by continuing the process of (ii).

The number of integers less than N and prime to it is often denoted by $\phi(N)$.

Examples XXXIV c.

1. If a be prime to b, then $a+b$ is either prime to b or else they have a common factor z.

2. Any square number is of the form $4m$ or $8m+1$.

3. The sum of a number and its square is always even.

4. If any square number be divided by 12, the remainder is also a square number.

5. Any cube number is of the form

(i) $9p$ or $9p \pm 1$; (ii) $4p$ or $4p \pm 1$; (iii) $7p$ or $7p \pm 1$.

6. Show that the product of any three consecutive numbers is divisible by $\lfloor 3$.

7. If N be odd and prime to 5, then $N^4 - 1 = M(80)$.

8. If n be any whole number, $n(n^2-1)(n^2-4)$ is divisible by 120.

9. If n be even, then $n^3 - 4n$ is a multiple of 48.

10. If N be even, then $N^2(N^2-4)$ is divisible by 192.

11. If N be a prime number greater than 3, prove that N^2-1 is divisible by 24.

12. Prove that the 5th power of any number is of one of the forms $11m$, $11m \pm 1$.

13. Prove that every square number prime to 6 is of the form $24n+1$.

14. If x be any prime number greater than 5, excepting 17, $x^{16}-1$ is divisible by 16320.

15. Find the sum of all the divisors of 24.

16. How many pairs of factors has the number 792?

17. Find the number of numbers less than 1904 and prime to it.

18. If $\phi(a)$ denotes the number of integers less than a and prime to a; prove that $\phi(ab) = \phi(a) \times \phi(b)$.

19. State and prove the converse of Wilson's Theorem.

20. State Fermat's Theorem. Deduce that $N^{(n-1)n} - 1$ is divisible by n^n, if n is prime and N prime to n.

21. Find how many different numbers will divide 25200 without remainder. Find their sum.

22. If a square number N equals $a^p b^q c^r \ldots$, how many pairs of factors has it?

23. Find the highest power of a number a contained in $n!$

$\left[\text{Denote greatest integer in } \frac{n}{a} \text{ by } I\left(\frac{n}{a}\right)\right]$.

Show that 100! is divisible by 2^{97}.

375. Example I. *Reduce* $\frac{1271}{4462}$ *to a continued fraction.*

$$\text{Fraction} = \frac{1271}{4462} = \frac{1}{\frac{4462}{1271}}$$

$$= \frac{1}{3+\frac{649}{1271}} = \frac{1}{3+\frac{1}{\frac{1271}{649}}}$$

$$= \frac{1}{3+\frac{1}{1+\frac{622}{649}}} = \frac{1}{3+\frac{1}{1+\frac{1}{1+\frac{27}{622}}}}$$

$$= \frac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{23+\frac{1}{27}}}}} \qquad \text{(A)}$$

Such a fraction is called a **continued fraction.**

To save space continued fractions are often expressed in the following form

$$\frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{23+}\ \frac{1}{27} \qquad \text{(B)}$$

It will be noticed that the process by which the vulgar fraction is reduced to the form A involves all the work of finding the G. C. M. of the numerator and denominator; but whereas in finding the G. C. M. there is no need to put down the quotients, in reducing to a continued fraction it is the quotients that are required.

The work may be arranged as in the following example.

Example II. *Express* $\frac{749}{3223}$ *as a continued fraction.*

2nd Quotient	3	749	3223	4	1st Quotient.
	2	68	227	3	
	22	22	23	1	
			1		

Quotients 4 3 3 2 1 22

Continued Fraction $\frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{3+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{22}$.

The successive quotients are called **partial quotients**, the **complete quotient** at any stage being a mixed number; thus corresponding to the partial quotient 2 in Example II the complete quotient is $2\frac{22}{23}$.

376. Convergents to a continued fraction.

The fraction obtained by stopping at any partial quotient is called a **convergent** to the fraction.

In Example II, the convergents are

$$\frac{1}{4} = \frac{1}{4},$$

$$\frac{1}{4+}\ \frac{1}{3} = \frac{3}{3\times4+1} = \frac{3}{13},$$

$$\frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{3} = \frac{3\frac{1}{3}}{(3\times4+1)+\frac{1}{3}\times4} = \frac{3\times3+1}{3\times13+4} = \frac{10}{43},$$

$$\frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{3+}\ \frac{1}{2} = \frac{(3\times3+1)+\frac{1}{2}\times3}{(3\times13+4)+\frac{1}{2}\times13} = \frac{2\times10+3}{2\times43+13} = \frac{23}{99},$$

$$\frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{3+}\ \frac{1}{2+}\ \frac{1}{1} = \frac{(2\times10+3)+\frac{1}{1}\times10}{(2\times43+13)+\frac{1}{1}\times43} = \frac{23\times1+10}{99\times1+43} = \frac{33}{142},$$

and complete continued fraction

$$= \frac{(23+10)+\frac{1}{22}\times23}{(99+43)+\frac{1}{22}\times99} = \frac{22\times33+23}{142\times22+99} = \frac{749}{3223}.$$

Convergents are $\frac{1}{4}, \frac{3}{13}, \frac{10}{43}, \frac{23}{99}, \frac{33}{142}, \frac{749}{3223}$.

377. Law of formation of convergents to a continued fraction.

Let the fraction be $\frac{1}{a_1+}\ \frac{1}{a_2+}\ \frac{1}{a_3+}\ \dots\ \frac{1}{a_n}$; and let $\frac{p_r}{q_r}$ be the convergent corresponding to the partial quotient a_r; then we are going to show that

$$p_r = a_r p_{r-1} + p_{r-2} \quad \text{and} \quad q_r = a_r q_{r-1} + q_{r-2}.$$

Suppose this law does hold for a particular value r, so that

$$\frac{p_r}{q_r} = \frac{a_r p_{r-1} + p_{r-2}}{a_r q_{r-1} + q_{r-2}}.$$

* This fraction is derived from the fraction above it by changing 3 into $3\frac{1}{3}$.

The $\overline{r+1}^{\text{th}}$ convergent differs from the r^{th} in having $a_r + \frac{1}{a_{r+1}}$ in place of a_r; hence

$$\frac{p_{r+1}}{q_{r+1}} = \frac{\left(a_r + \frac{1}{a_{r+1}}\right)p_{r-1} + p_{r-2}}{\left(a_r + \frac{1}{a_{r+1}}\right)q_{r-1} + q_{r-2}} = \frac{a_r p_{r-1} + p_{r-2} + \frac{p_{r-1}}{a_{r+1}}}{a_r q_{r-1} + q_{r-2} + \frac{q_{r-1}}{a_{r+1}}}.$$

$$= \frac{p_r + \frac{p_{r-1}}{a_{r+1}}}{q_r + \frac{q_{r-1}}{a_{r+1}}} = \frac{a_{r+1}p_r + p_{r-1}}{a_{r+1}q_r + q_{r-1}}.$$

Therefore if the law holds for any convergent, it holds for the next.

By actually working out the convergents it is found that

$$\frac{p_1}{q_1} = \frac{1}{a_1}, \quad \frac{p_2}{q_2} = \frac{a_2}{a_1 a_2 + 1}, \quad \frac{p_3}{q_3} = \frac{a_3 a_2 + 1}{a_3(a_1 a_2 + 1) + a_1}.$$

Hence the law holds when $r = 3$, and, therefore, for any value greater than 3.

Example. *Write down the convergents to* $\frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{5}$.

Convergents are $\frac{1}{1}$, $\frac{2}{3}$, $\frac{7}{10}$, $\frac{23}{33}$, $\frac{99}{142}$, $\frac{518}{743}$.

378. Properties of convergents.

(i) The convergents are alternately greater and less than the continued fraction.

For $a_1 <$ whole denominator; $\therefore \frac{1}{a_1}$ is greater than the fraction;

$$\frac{1}{a_2} > \frac{1}{a_2 + \frac{1}{a_3+}}; \quad \therefore \frac{1}{a_1+}\ \frac{1}{a_2}$$ is less than the fraction,

and so the argument continues.

(ii) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

For $p_n q_{n-1} - p_{n-1} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(a_n q_{n-1} + q_{n-2})$

$$= p_{n-2} q_{n-1} - p_{n-1} q_{n-2}$$

$$= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}).$$

Similarly $p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = -(p_{n-2} q_{n-3} - p_{n-3} q_{n-2})$.

$\therefore$ $(p_n q_{n-1} - p_{n-1} q_n) = (-1)^{n-2}(p_2 q_1 - p_1 q_2)$

$$= (-1)^{n-2}(a_1 a_2 - a_1 a_2 - 1)$$

$$= (-)^{n-1}.$$

(iii) Each convergent is in its lowest terms.

For any common factor of p_r and q_r would divide $p_r q_{r-1} - p_{r-1} q_r$; that is, would divide ± 1, which is impossible.

$a \sim b$ means the difference between a and b, i.e. is $a-b$ or $b-a$, according as $a > b$ or $b > a$.

(iv) Each convergent is nearer to the continued fraction than any preceding convergent.

Let $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}$ be three successive convergents; then if x denote the continued fraction, and k the complete quotient after the $(n)^{\text{th}}$ partial quotient,

$x = \frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}$ since x differs from $\frac{p_{n+1}}{q_{n+1}}$ in taking the complete quotient k instead of the partial quotient a_{n+1}.

$$\therefore \frac{p_n}{q_n} \sim x = \frac{p_n q_{n-1} \sim p_{n-1} q_n}{q_n (kq_n + q_{n-1})} = \frac{1}{q_n (kq_n + q_{n-1})},$$

Similarly $x \sim \frac{p_{n-1}}{q_{n-1}} = \frac{k(p_n q_{n-1} \sim p_{n-1} q_n)}{q_{n-1}(kq_n + q_{n-1})} = \frac{k}{q_{n-1}(kq_n + q_{n-1})}$.

Now $k > 1$ and $q_n > q_{n-1}$; $\therefore \frac{k}{q_{n-1}(kq_n + q_{n-1})} > \frac{1}{q_n (kq_n + q_{n-1})}$.

Hence $\frac{p_n}{q_n}$ is nearer to x than is $\frac{p_{n-1}}{q_{n-1}}$.

Similarly $\frac{p_{n-1}}{q_{n-1}}$ is nearer to x than $\frac{p_{n-2}}{q_{n-2}}$.

And the theorem follows.

379. To reduce a quadratic surd to a continued fraction. An example will render the process clear.

$$\sqrt{7} = 2 + \sqrt{7} - 2 = 2 + \frac{(\sqrt{7}-2)(\sqrt{7}+2)}{\sqrt{7}+2}$$

$$= 2 + \frac{3}{\sqrt{7}+2}$$

$$= 2 + \frac{1}{\frac{\sqrt{7}+2}{3}} \qquad \text{(A)}$$

(The greatest integer in $\sqrt{7}+2$ is 4, hence greatest integer in $\frac{\sqrt{7}+2}{3}$ is 1.)

$$\frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} = 1 + \frac{2}{\sqrt{7}+1}$$

$$= 1 + \frac{1}{\frac{\sqrt{7}+1}{2}} \qquad \text{(B)}$$

$$\frac{\sqrt{7}+1}{2} = 1+\frac{\sqrt{7}-1}{2} = 1+\frac{3}{\sqrt{7}+1}$$

$$= 1+\frac{1}{\frac{\sqrt{7}+1}{3}} \qquad \text{(C)}$$

$$\frac{\sqrt{7}+1}{3} = 1+\frac{\sqrt{7}-2}{3} = 1+\frac{1}{\sqrt{7}+2} \qquad \text{(D)}$$

$$\sqrt{7}+2 = 4+\sqrt{7}-2 = 4+\frac{1}{\frac{\sqrt{7}+2}{3}} \quad \text{from (A).}$$

(And steps *B*, *C*, *D* follow as before.)

Hence $$\sqrt{5} = 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+}}}} \quad \&c.$$

and the partial quotients 1, 1, 1, 4 continually repeat.

Hence $\sqrt{5}$ equals a recurring continued fraction.

380. To show that a recurring continued fraction equals one of the roots of a quadratic equation with rational coefficients.

Numerical example. *Find the value of*

$$\frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{4+} \text{ to infinity.}$$

Let $$x = \frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{4+} \text{ to infinity;}$$

then $$x = \frac{1}{3+}\ \frac{1}{4+x}$$

$$= \frac{4+x}{13+3x};$$

i.e. $3x^2+12x-4 = 0$.

Hence the fraction is the positive root of this equation.

General Proof.

Let $$x = a+\frac{1}{b+}\ \frac{1}{c+}\cdots\frac{1}{h+}\ \frac{1}{k+}\ \frac{1}{y},$$

and let $$y = \frac{1}{m+}\ \frac{1}{n+}\cdots\frac{1}{u+}\ \frac{1}{v+}\ \frac{1}{y}.$$

Let $\frac{p}{q}$, $\frac{p'}{q'}$ be the convergents corresponding to h, k respectively.

Then $$x = \frac{p'y+p}{q'y+q}.$$

Solve this for y, $$\cdots \quad y = \frac{p-qx}{q'x-p'}.$$

If $\frac{r}{s}$, $\frac{r'}{s'}$ are the convergents to y corresponding to u, v, respectively,

$$y = \frac{r'y+r}{s'y+s}.$$

Substitute for y the value already obtained;

$$\frac{p-qx}{q'x-p'} = \frac{r'(p-qx)+r(q'x-p')}{s'(p-qx)+s(q'x-p')}$$

which is a quadratic equation.

381. Application of continued fractions to the solution of indeterminate equations.

Example. *Find the general solution of* $293x-372y=7$.

Find the nearest convergent to $\frac{293}{372}$.

3	293	372	1
2	56	79	1
3	10	23	2
	1	3	3

Quotients 1 3 1 2 2 3 3.

Convergents $\frac{1}{1}$ $\frac{3}{2}$ $\frac{4}{5}$ $\frac{11}{14}$ $\frac{26}{33}$ $\frac{89}{113}$ $\frac{293}{372}$.

Hence $293\times113-372\times89=1$.

Multiply by 7. $293\times791-372\times623=7$.

Subtract from given equation $293(x-791)=372(y-623)$;

$$\therefore \quad \frac{x-791}{372} = \frac{y-623}{293}.$$

$$\frac{x-47}{372}-2 = \frac{y-87}{293}-2$$

$$\frac{x-47}{372} = \frac{y-87}{293} = k \text{ (suppose)};$$

$$\therefore \quad x = 372k+47$$

$$y = 293k+87.$$

Examples XXXIV d.

1. Prove that the difference between two successive convergents to the continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots$$

is a fraction whose numerator is unity.

2. In the continued fraction $\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots$ show that

$$p_{n+2}-(ab+2)\,p_n+p_{n-2}=0,\quad q_{n+2}-(ab+2)\,q_n+q_{n-2}=0.$$

3. Prove that the n^{th} convergent to the continued fraction $\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}$ is $\frac{n}{n+1}$.

4. Find the convergents to $\frac{4129}{14256}$.

5. Find the value of the infinite continued fraction

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots.$$

6. If $x=a+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots$, $y=b+\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots$, both fractions being continued to infinity, prove that

$$ay=bx=xy-1.$$

7. If p_3, p_4 are the numerators of the 3rd and 4th convergents to $a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4}$, prove that

$$\frac{p_4}{p_3}=a_4+\frac{1}{a_3+}\,\frac{1}{a_2+}\,\frac{1}{a_1}.$$

8. If $\frac{p_n}{q_n}$ denotes the n^{th} convergent to the continued fraction $\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$, prove that

$$\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{(-1)^{n-1}}{q_n q_{n-1}}.$$

9. Verify that the roots of the equation $x^2-3x+1=0$ can be written in the forms

$2+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}$ to infinity, and $1-\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}$ to infinity.

10. Express $\frac{4+\sqrt{19}}{3}$ as a periodic continued fraction, and verify the result.

11. Express the roots of $x^2-2x-7=0$ as continued fractions, and find the first convergent to the positive root which differs from it by less than $\frac{1}{100}$.

12. Show that all convergents to $a+\frac{1}{b+}\frac{1}{c+}\ldots$ are in their lowest terms.

Express $\sqrt{\frac{5}{3}}$ as a continued fraction and find the first three convergents.

13. Find a quadratic equation one root of which is equal to

$$1+\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\ldots.$$

14. Show that

$$\sqrt{a^2+4a}=a+\frac{2}{1+}\frac{1}{a+}\frac{1}{1+}\frac{1}{a+}\frac{1}{1+}\ldots.$$

15. Prove that

$$\sqrt{9a^2+3}=3a+\frac{1}{2a+}\frac{1}{6a+}\frac{1}{2a+}\frac{1}{6a+}\ldots.$$

16. Find the value of $1+\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\ldots.$

17. Find all the convergents to 3·1416.

18. If $\frac{p_{2n}}{q_{2n}}$ is the $2n^{\text{th}}$ convergent to $\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\ldots$, prove that $p_{2n}=p_{2n-2}+2q_{2n-2}$.

Find the general solution of the following equations:

19. $37x+35y=1$.

20. $317x-412y=3$.

21. $11x+9y=67$.

22. $74x-89y+3=0$.

MISCELLANEOUS EXAMPLES

1. Find the Highest Common Factor of
$x^3-3x-28$, x^3-13x^2+42x, and $x^3-(7+2b^2)x^2+(14+b)b^2x-7b^2$.

2. Solve the equations:

(i) $\dfrac{1}{x}-\dfrac{1}{x-a-b}=\dfrac{1}{a}+\dfrac{1}{b}$;

(ii) $\begin{aligned} x^2+y+z &= 21 \\ x^2+yz &= 22 \\ x^2yz &= 96 ; \end{aligned}$

(iii) $\sqrt{x^2+x+4}+\sqrt{x^2-x+3}=2x+1$.

3. If a, b, c, d are unequal, and if

$$\sqrt{(a^2+b^2)(c^2+d^2)}-\sqrt{(a^2+c^2)(b^2+d^2)}=(b-c)(d-a),$$

then a, b, c, d are in proportion.

4. A tailor buys a piece of cloth for a certain sum and sells two yards more than one-fourth of the whole at a profit of 25 per cent. He afterwards succeeds in selling the rest at a profit of 75 per cent., and finds he made a profit of 60 per cent. on the whole transaction. How many yards of cloth did he buy?

5. Define the Arithmetic, Geometric, and Harmonic means between two quantities.

There are four numbers, the first three of which are in Arithmetic progression and the last three in Harmonic progression. Show that the four numbers are in proportion.

6. Show that $\phi(x)\equiv\dfrac{\log(\log x)}{\log r}$ is a function of x which satisfies the condition $\phi(x^r)\equiv 1+\phi(x)$.

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7. Simplify

(i) $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}$;

(ii) $\dfrac{(a+b+c)^2}{a^2-(b+c)^2}\times\dfrac{(a-b)^2-c^2}{(c+a)^2-b^2}$.

8. Find the square root of

(i) $9x^6-6x^5-11x^4+10x^3+2x^2-4x+1$;

(ii) $62-20\sqrt{6}$.

9. Solve the equations:

(i) $x^2-10x+18=0$;

(ii) $2x^2+3xy+4y^2=62$, $x^2+xy+y^2=19$.

10. *A* and *B* start to run to a flag 500 yards off and back; *A*, returning, meets *B* 50 yards from the flag and arrives at the starting-point 40 seconds before *B*. How long did *A* take to run the whole distance?

11. It is desired to make with the least amount of sheet metal a closed cylindrical tin to contain exactly 1000 cubic centimetres of water. Plot a curve which will show the relation between the amount of metal in square centimetres and the diameter of the tin. Use this curve to determine the required diameter. Give the equation of the curve.

12. Show that $\left\{\frac{(a^{\frac{1}{2}}+b^{-\frac{1}{2}})^4-(b^{\frac{1}{2}}+a^{-\frac{1}{2}})^4}{a^2-b^2}\right\}^{\frac{1}{2}}=1+\frac{1}{(ab)^{\frac{1}{2}}}$.

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13. Determine whether

$$\{(m+n)(x+a)\}^2-(mx+na)^2-(nx+ma)^2-2(mx+na)(nx+ma)=0$$

is an identity or an equation.

14. Resolve into factors:

(i) $\frac{1}{2}\{(a+b+c)^5-a^5-b^5-c^5\}$;

(ii) $a^4+b^4+c^4-2b^2c^2-2c^2a-2a^2b^2$;

(iii) $(a^2+2bc)^3+(b^2+2ca)^3+(c^2+2ab)^3$ Hall, §525.
$-3(a^2+2bc)(b^2+2ca)(c^2+2ab)$.

15. Solve graphically the equation $6x^2-13x-80=0$, and verify the result by calculation.

16. Explain the special value of the rule for rationalizing the denominator of a compound surd.

Assuming that $\sqrt{3}=1{\cdot}732$, find the value of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$.

17. Prove the formulae for finding (i) the sum of n terms of an Arithmetic Progression; (ii) the sum of n terms of a Harmonic Progression.

Find the sum of

(i) $-7-4-1+2\ldots$ to 17 terms.

(ii) $\log a+\log\left(\frac{a^2}{b}\right)+\log\left(\frac{a^3}{b^2}\right)+\ldots$ to n terms.

If $a=25$, $b=2$, and $n=100$, calculate to the nearest integer the sum of the series.

18. Find two successive terms in the expansion of $(1+x)^{29}$ which are such that their coefficients are in the ratio 5 : 1.

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19. Obtain a meaning for a^{-m} and a^0.

Divide $x^{\frac{5}{2}}-x^2-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}}$ by $x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2$.

Simplify $\sqrt{12}+\frac{1}{10}\sqrt{75}+6\sqrt{\frac{1}{12}}$.

20. Extract the fourth root of

$$x^8-\frac{3a^2x^7}{b}+\frac{27a^4x^6}{8b^2}-\frac{27a^6x^5}{16b^3}+\frac{81a^8x^4}{256b^4}.$$

21. Prove that in a quadratic equation $x^2+px+q=0$, the sum of the roots is $-p$ and their product is q.

B and C each attempt the same equation. B after reducing has only a mistake in the numerical term and finds for its roots $+8$ and $+2$. C after reducing has only a mistake in the coefficient of x, and finds for roots -9 and -1. Find the correct roots of the equation.

22. If a, b, c, d be in A.P.; a, e, f, d in G.P.; a, g, h, d in H.P. respectively; then $ad=ef=bh=cg$.

23. On a certain steamer the expense of coaling per mile is found to be directly proportional to the square of the speed maintained, being 2*s.* 6*d.* per mile when the speed is 30 miles per hour, the other expenses being £1 16*s.* 0*d.* per hour at all speeds. Express in pounds the expenses of a journey of 100 miles at x miles an hour; draw a diagram showing the expense of a journey of 100 miles at any speed from 0 to 45 miles an hour, and determine the most economical speed and the expense at that speed.

24. Express in rational form the condition that the equations $ax^2+bx+c=0$, $a'x^2+b'x+c'=0$ may have a common root.

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25. If $a+\sqrt{b}=c+\sqrt{d}$, neither b nor d being a perfect square, prove that $a=c$, $b=d$. §196.

Find the square root of $174+36\sqrt{6}$.

26. Solve (i) $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$;

(ii) $11x+7y+63xy=0=x+y+5xy$.

27. Draw the graph of $y=\frac{x^2-3x+4}{x^2+3x+4}$ for real values of x; and find algebraically the limits between which y must lie.

28. Find the square root of the ratio of two numbers whose arithmetic mean is twice their geometric mean. §239.

29. What is, in general, the meaning of $f(x)$.

If $f(x)\equiv x+\frac{1}{x}$, show that $f(x)=-f(-x)$. Verify this when $x=1$ and $x=0$. Comment on the results.

Draw the graph of $y=x+\frac{1}{x}$.

30. If $(1+x)^n=c_0+c_1x+c_2x^2\ldots+c_nx^n$ where n is a positive integer, prove that

$$\frac{2^{n+1}-1}{n+1}=\frac{c_0}{1}+\frac{c_1}{2}+\frac{c_2}{3}\ldots+\frac{c_n}{n+1}.$$

31. If
$$x+y+z=a$$
$$x^2+y^2+z^2=b$$
$$x^3+y^3+z^3=c,$$

find an equation in k whose roots are x, y, z.

32. Define the logarithm of a number to a given base; also define its characteristic and its mantissa. Give a single rule for determining the characteristic of the common logarithm of any number. Explain why negative numbers have no real logarithms.

33. Between one census and the next the native population of a town increased by 8 per cent., while the number of foreigners decreased from 200 to 150. The increase in the total population was 7 per cent. What was the total population at the second census?

34. If $a^{\frac{p}{q}}$ is defined as another form of $\sqrt[q]{a^p}$, prove that

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{1}}.$$

If $a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}} = 0$, prove that

$$a^3 + b^3 + c^3 + 3(a+b+c)(bc+ca+ab) = 30abc.$$

35. Sum $\dfrac{1}{\sqrt{2}+\sqrt{1}} + \dfrac{1}{\sqrt{3}+\sqrt{2}} + \dfrac{1}{\sqrt{4}+\sqrt{3}} + \ldots$ to n terms.

If $m, a_1, a_2, \ldots a_r, n$ be in A. P. and $m, h_1, h_2, \ldots h_r, n$ be in H. P., prove that $a_s h_{r-s+1} = mn$.

36. Quelle est la condition pour que dans l'équation

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0$$

le produit de deux racines soit égal au produit de deux autres ?

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37. Disposer de λ et μ de façon que le polynôme

$$x^4 - 4x^3 + x^2 + \lambda x + \mu$$

soit carré parfait et extraire la racine.

38. (i) Find the factors of $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$.

(ii) Simplify $\left(x+\frac{1}{x}\right)^2 + \left(y+\frac{1}{y}\right)^2 + \left(xy+\frac{1}{xy}\right)^2$

$$- \left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(xy+\frac{1}{xy}\right).$$

39. Solve (i) $\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{2}{x-3}$;

(ii) $\dfrac{x}{a} + \dfrac{a}{x} = \dfrac{y}{b} + \dfrac{b}{y} = \dfrac{x}{y} + \dfrac{y}{x}$.

40. Find the number of permutations of n things taken all together of which r are alike. If this number be denoted by p_r show that $(p_3 - p_2)(p_4 - p_3) \ldots (p_{n-1} - p_n) = \dfrac{p_2 p_3 \ldots p_{n-1} p_n}{p_2 p_{n-1}} \times \dfrac{P_1 P_n}{P_2}$.

41. Berechne logarithmisch das 18te Glied der geometrischen Reihe, in welcher das Anfangsglied a und der konstante Quotient q die folgenden Werte haben: $a = 3$, $q = \frac{2}{3}$.

42. A gentleman arrives at the railway station nearest to his house an hour and a half before the time at which he had ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles an hour, and, meeting his carriage when it had travelled 2 miles, reaches home exactly an hour earlier than he had originally expected. How far is his house from the station and at what rate was his carriage driven?

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43. Explain the use of brackets; and prove that

$$a+(b-c) = a+b-c$$
$$a-(b-c) = a-b+c.$$

Give any geometrical representation of the fact that the product of two negative numbers is positive.

44. A solid cube of lead weighs 126·44 lb.; 998 ounces of water occupy one cubic foot, and a cubic foot of lead is 11·352 times as heavy as a cubic foot of water. Find by logarithms the length of a side of the cube of lead correctly to four significant figures.

45. Prove that the arithmetic mean of n positive quantities is greater than their geometric mean.

Find the minimum value of $2x+3y+4z$ subject to the condition that $x^2y^3z^4 = c^9$ where c is constant.

46. Entwickle mit Hilfe des binomischen Lehrsatzes:

$$(x-2y+1)^5.$$

47. Transform 1022634 from the septenary to the denary scale, and 124·96 from the denary scale to that with radix 5.

48. Trouver la valeur limite de $\sqrt{2+\sqrt{2+\sqrt{2+}}}$, le nombre des radicaux étant infini.

Cf. Chr. i. 402-34.

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49. Plot the graph of the expression $\frac{(x-2)(x+1)}{2x-1}$ between the values $x = -3$ and $x = 4$, and use it to obtain approximate solutions of the equation $x^2-3x-1 = 0$,

of $x^2+x-3=0$, or

$x^2+3x-4=0$.

50. Solve: (i) $\frac{x-1}{x} - 8\frac{x}{x-1} = 2$;

(ii) $x^2 - xy = 5,\ x + 3y = 11.$

51. Trouver la somme des cubes des n premiers nombres impairs.

52. If x and y be the roots of the equation in λ

$$\lambda^2 + 1 = a(\lambda + 1),$$

prove that $\frac{x^3}{x+1} + \frac{y^3}{y+1} = a^2 + \frac{a}{2} - 1.$

53. Prove that if $\frac{x}{y} = \frac{u}{v}$, then either of these fractions is equal to $\frac{x+au}{y+av}$. Hence show that if $\left(\frac{a+b}{c+d}\right)^2 = \frac{ab}{cd}$, then either $\frac{a}{b} = \frac{c}{d}$ or $\frac{b}{a} = \frac{c}{d}$.

54. Expand $\frac{2x^2 - 3x + 4}{(1-x)(2-x)(2+3x)}$ in positive integral powers of x, giving the general term.

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55. Find all the values of x for which

$$4 < \frac{x+14}{x^2-1} < 10.$$

56. Solve (i) $x^2(x^2 - 3ax - a^2) = a^2(x^2 + 3ax - a^2)$;

(ii) $a^{(x+y)\log b} + b^{(x+y)\log a} = ab.$

$$(ab)^{xy \log a \log b} = a + b.$$

57. If 1, ω, ω^2 represent the cube roots of unity, show that

(i) $(x - \omega y)(x - \omega^2 y) = x^2 + xy + y^2.$

(ii) $a^3 + b^3 + c^3 = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$; $+3abc$.

58. The series $a_0 + a_1 x + a_2 x^2 + \ldots$ is such that

$$a_n - 3a_{n-1} + 2a_{n-2} = 0.$$

Prove that its general term is $(A + B.2^n)x^n$; and determine A and B in terms of a_0 and a_1.

59. Find a quadratic equation one root of which is

$$1+\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\cdots;$$

and express the other root as a continued fraction.

60. (i) Quelle est la limite de

$$\frac{1+2+3+4\ldots+(n+1)}{1+3+5+7\ldots+(2n+1)} \text{ pour } n \text{ infini ?}$$

(ii) On considère le déterminant :

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

montrer *a priori* qu'il est nul.

EXAMINATION PAPERS

OXFORD UNIVERSITY.

RESPONSIONS. SEPTEMBER, 1907.

1. If $x = 2a^2 - b^2 - c^2$,
$y = a^2 + 2b^2 + c^2$,
$z = a^2 - b^2 + 2c^2$,
find the value of $4x - 2y + 3z$,
when $a = 4$, $b = -3$, $c = 2$.

2. Subtract $a^3 - 2x(a + 3x)$ from $(a + 2x)(a - 3x)$, and multiply $a^3 - a^2 + a - 1$ by $a^3 + a^2 - a + 1$.

3. Find the G. C. M. of
$$8x^3 + 6x^2 - 4x - 3 \text{ and } 12x^3 + 5x^2 + x + 3,$$
and the L. C. M. of
$$x^2 - x - 6 \text{ and } x^2 - 2x - 3.$$

4. Simplify:

(1) $\dfrac{3a - 2b}{ab} + \dfrac{4b}{a^2 + 2ab} - \dfrac{3a}{ab + 2b^2}$;

(2) $\dfrac{x - \dfrac{y^2}{x}}{\dfrac{x^2}{y^2} - \dfrac{y^2}{x}} \times \left(\dfrac{1}{x^2} + \dfrac{1}{y^2}\right)$.

5. Extract the square root of
$$16x^4 - 24x^3y + 49x^2y^2 - 30xy^3 + 25y^4.$$

6. Prove that if any number be divided into any two parts, the difference of the squares of the parts is equal to the number multiplied by the difference of the parts.

7. Solve the equations:

(1) $\dfrac{7x - 4}{15} - 2 + \dfrac{3x + 3}{16} = \dfrac{7x + 1}{20}$;

(2) $\dfrac{ac}{bx} - \dfrac{a}{2} = \dfrac{bc}{ax} + \dfrac{b}{2}$.

8. (1) Solve the equation:

$$12x+13y=74,$$
$$17x-19y=30.$$

(2) If $\frac{x}{2}+y=9$, and $\frac{x}{6}=y-1$,

find the value of $x-y$.

9. Find two consecutive numbers such that the sum of the third and fourth parts of the lesser number shall be equal to the sum of the half and the fortieth part of the greater.

10. A man invests £50,000 partly in Railway Shares and partly in Colonial Bonds. From the former he gets 3 per cent. on the sum invested, from the latter 3½ per cent., and from the first investment he derives £200 a year more than from the second. What was the amount invested in each case?

Responsions. Michaelmas Term, 1907.

1. Multiply together

$$(a+b)c,\ a+c,\ \text{and}\ a-b,$$

and find the value of the result when

$$a=3,\quad b=-2,\quad c=4.$$

2. Add together

$$a^3-2b(a+3b),\quad (a+2b)(a-3b),\quad \text{and}\quad 3b(a+4b)-2a^2$$

Subtract

$$\frac{3x-6y}{3}+a \text{ from } x-\frac{4y-2c}{2}.$$

3. Find the G. C. M. of

$$2x^3-x-1 \text{ and } x^4-x^3-2x+2,$$

and the L. C. M. of

$$a^2+5ab+6b^2,\quad a^2+ab-2b^2,\quad \text{and}\quad a^2+2ab-3b^2.$$

4. Simplify:

(1) $$\frac{a^2+4b^2+\dfrac{15b^4}{a^2-4b^2}}{a+2b+\dfrac{3b^2}{a-2b}};$$

(2) $$\left(\frac{1}{y^2}-\frac{1}{x^2}\right)\times\frac{x+\dfrac{y^2}{x}}{\dfrac{x^3}{y^2}-\dfrac{y^2}{x}}\div\frac{1}{x^2}.$$

5. Extract the square root of

$$4x^6 - 12x^5y + 17x^4y^2 - 12x^3y^3 + 4x^2y^4.$$

6. Prove that the square of the sum of any two numbers together with the square of their difference is equal to twice the sum of the squares of the two numbers.

7. Solve the equations:

(1) $$\frac{\frac{1}{x} - 3}{\frac{}{x} - 2} = \frac{1}{4};$$

(2) $$\frac{x-a}{3} - a = \frac{x-b}{5} - 3b.$$

8. (1) Solve the equations:

$$\frac{10x}{9} + \frac{9y}{2} = 56,$$

$$\frac{3y}{10} - \frac{2x}{3} = \frac{2}{5};$$

(2) If $2y - \frac{x-3}{5} = 4$, and $3x + \frac{y-2}{3} = 9$,

find the value of $\frac{x+y}{x-y}$.

9. A tourist starts for a walk at noon and allows himself seven hours to reach a certain town. After walking two-thirds of the distance at $3\frac{1}{2}$ miles an hour, he rests for an hour and a quarter, and then finds that he must walk four miles an hour if he is to arrive at the town by seven o'clock. What was the length of his walk?

10. A man's income amounts to £1500 a year. On this he pays £63. 10*s.* income tax, partly at the rate of 1*s.* in the £ and partly at ninepence.

Determine the amount on which he pays at each of the two rates.

CAMBRIDGE UNIVERSITY.

Previous Examination.

Wednesday, December 11, 1907.

1. If the square of half the difference between a given number and its square be subtracted from the square of half the sum of the same number and its square, show that the result is the cube of the given number.

2. Find the difference between

$$(x-1)^3+3x^3(x-1) \text{ and } x^3+3x(x-1)^3.$$

Divide $x^5-29x+12$ by x^3+2x-1.

3. Simplify: (1) $b-2a-\{c-a-(b-a+c)\}$,

$$(2)\quad \frac{x^3}{(x-y)(x-z)}+\frac{y^3}{(y-z)(y-x)}+\frac{zx}{(z-x)(z-y)}.$$

4. Find the Highest Common Factor of

$$2x^4-6x^3-2x^2-6x-4 \text{ and } 2x^3-5x^2-7x-2.$$

5. Solve the following:

$$(1)\quad \frac{x-a}{b}+\frac{x+b}{a}=\frac{a+b}{b},$$

$$(2)\quad 4x+15y=\tfrac{1}{3},\quad 5y-\tfrac{1}{2}x=\tfrac{1}{3},$$

(3) $5x^2-9x=5$, giving the answer to two decimals,

$$(4)\quad 7y^2+5xy=468,\quad 5x+4y=-3.$$

6. A father's age is three times his son's age; prove that, when the son is twice as old as he is now, the father will be twice as old as the son.

7. Prove that $(x^m)^n=x^{mn}$, when m and n are positive integers.

Simplify $(a^3bc^{-2})^{\frac{1}{2}}\times(a^{-1}b^{-3}c^2)^{\frac{1}{2}}$.

8. A number consists of three digits of which the middle one is equal to the sum of the other two; the square of the middle digit falls short of five times the product of the other two by 1, and 5 times the first digit equals twice the third digit. Find the number.

9. Find the sum of n terms of a geometrical progression whose first term is a and common ratio r.

Sum the series:

(1) $14{\cdot}5+15{\cdot}3+16{\cdot}1+\ldots$ to 20 terms,

(2) $24+36+54+\ldots$ to 8 terms.

10. Prove that $\log\frac{a}{b}=\log a-\log b$.

Find by logarithms $\sqrt[3]{27{\cdot}33}$, $(\cdot07925)^4$.

11. Draw the graph of

$$y=\frac{1}{3}\left\{\frac{30}{x-5}+23-2x\right\},$$

for values of x between 0 and 5. Find from the graph the values of x and y when $y=2x$.

LONDON UNIVERSITY MATRICULATION EXAMINATION.

SEPTEMBER 10, 1907.

1. 2. 3. Arithmetic.

4. (1) Simplify $\dfrac{\frac{1}{x}+\frac{2}{y}}{\frac{1}{x}-\frac{2}{y}}-\dfrac{\frac{1}{x}-\frac{2}{y}}{\frac{1}{x}+\frac{2}{y}}$.

(2) Solve the equations

$$ax+by=a^2+2ab-b^2$$

$$bx+ay=a^2+b^2.$$

5. A bill for 10 lb. of butter, 7 fowls and 52 quarts of milk amounts to 42*s*. 7*d*., when a bill for 1 lb. of butter, 2 fowls, and 4 quarts of milk is 6*s*. 10*d*. The next week, butter having risen 1*d*. a lb. and fowls 3*d*. each, the bill for 8 lb. of butter, 5 fowls, and 46 quarts of milk is 35*s*. 11*d*. Find the prices for the first week.

6. *A* and *B* make from the same place a journey of 56 miles; *A* starts 3 hours and 10 minutes after *B*, but travels 5 miles an hour faster. They arrive at the same time. Find the pace of each.

7. From the following data draw a graph representing the time of sunrise, and estimate the time of rising on Feb. 15th, May 16th, and on the day of earliest rising:—

Date	Time h. m.	Date	Time h. m.
Jan. 1	8 8	May 31	3 51
Jan. 30	7 44	June 30	3 48
March 2	6 47	July 30	4 21
April 1	5 39	Aug. 29	5 8
May 1	4 35		

8. Trace, with the same axes, graphs of

(1) $y = 0{\cdot}3x^3 - 1{\cdot}2$;

(2) $y = x^3$.

With the help of the graphs, solve the equation

$$x^3 = 0{\cdot}3x^2 - 1{\cdot}2.$$

9. There are 2457 plants in a strawberry bed; in each row the number of plants exceeds the number in the preceding row by a fixed amount; in the first row there are 77, and in the last row 157 plants. How many rows are there? If you use a formula, prove it.

10. The first term of a Geometrical Progression is $\frac{a^5}{x^3}$, the fourth term is $\frac{x}{a}$. Find the sum of the first 21 terms. If you use a formula, prove it.

JANUARY, 1908.

1. Arithmetic.

2. Water flows into a cistern from a tap *A*, and flows away through a tap *B*. When both *A* and *B* are open the cistern is filled in p minutes, and when only *B* is open the cistern is emptied in q minutes. In how many minutes can the cistern be filled when *B* is closed?

3 and 4. Arithmetic.

5. (*a*) Find the factors of

(i) $x^2+x(b+c)-a^2-a(b+c)$;

(ii) $x^4-7x^2y^2+y^4$.

(*b*) Find to three significant figures the value of

$$\left(\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}\right)^2.$$

6. When a train is running at a speed of 45 miles per hour against the wind the air pressure on the front of the engine is 12·675 pounds per square foot. Assuming that the pressure is ·003 V^2 pound per square foot where V is the velocity of the train relative to the wind, measured in miles per hour, find the rate at which the wind is blowing, and the pressure when the train is going at 45 miles per hour with the wind.

7. With the same axes draw the graphs of

(i) $y=3x^2-4x+2$ and (ii) $y=6x+12$.

Find the values of x at the points where the graphs intersect, and verify by solving the equation

$$3x^2-4x+2=6x+12.$$

8. A besieged fortress is held by 5700 men who have provisions for 66 days. If the garrison loses 20 men each day, for how many days can the provisions hold out?

9. A man at the beginning of each of ten successive years deposits £100 with a company which pays him interest at the rate of 4 per cent. per annum, and he allows his deposits to accumulate at compound interest. Given that $(1{\cdot}04)^{10}=1{\cdot}480234$, find to the nearest penny how much he has at the end of the tenth year.

10. Given that $x^4+4x^3+px^2+qx+9$ is the square of x^2+ax+b, find all the possible values of a, b, p and q.

SCOTTISH UNIVERSITIES JOINT BOARD

PRELIMINARY EXAMINATIONS.

MATHEMATICS. LOWER STANDARD. OCTOBER, 1906.

1–5. Arithmetic.

6. Find the value of $\sqrt{\left(\frac{3a^x+2b^y}{ax+by}\right)}$,

when $a=2,\ b=3,\ x=3,\ y=2.$

7. Express

$$\frac{x^2-a^2}{x^2+(d-a)x-da}+\frac{x^2-b^2}{x^2+(d-b)x-db}+\frac{x^2-c^2}{x^2+(d-c)x-dc}$$

as a single fraction at lowest terms.

Solve the equation $\frac{3}{x-1}-\frac{2}{2x-5}=\frac{2}{x+2}$.

8. Find an equation of the form $Ax+By-1=0$ where A and B are independent of x and y, which has the two solutions

$$x=1,\ y=-1, \text{ and } x=2,\ y=3.$$

Trace the graph of the equation.

9. How can you tell by examining the coefficients whether Ax^2+Bx+C can be resolved into factors, the coefficients of which are real and rational?

Factorize the following:

(i) $(ax+by)^2-(bx-ay)^2$; (ii) $2x^2-3x+1$.

10. Solve the following equations:

(i) $(2x+1)^2+4(2x+1)-5=0$;

(ii) $\frac{x-1}{x^2-5x+4}+\frac{x+1}{x+5}=\frac{8}{5}$.

11. A, B, C are three coaling stations on a parallel of latitude whose circumference is 3150 miles. The distance between A and B is 150 miles, but the distances BC and CA are unknown. A cruiser, starting from A, steams to B with a slow uniform speed. She coals at B, increases her speed and steams to C. At C she coals, again increases her speed in the same ratio as before and finally steams to A. If the time taken between each of the three stations be the same, find the distances BC and CA.

Mathematics. Intermediate and Higher.

First Paper. September, 1906.

1. Arithmetic.

2. Simplify the following:

$$(b-c)(x-a)^2+(c-a)(x-b)^2+(a-b)(x-c)^2.$$

Show that numerical values can be found for A, B, C, such that the equation

$$2x^2+3x+5 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

shall be an identity.

3. Solve the equations:

(i) $(a^2-b^2)(1-x^2) = 4abx$;

(ii) $(x^2-3x-5)(x^2-3x+3)+7 = 0$.

4. State and prove the condition that the roots of the equation

$$ax^2+bx+c = 0$$

may be real when a, b, and c are real.

Show that the roots of

$$(q^2-pr)x^2+2(p+r)x-4 = 0$$

are always real for real values of p, q, r.

Find the greatest value that $5-2x-3x^2$ can have for real values of x.

5. Show that when a, b, c, and d are in proportion $\dfrac{a-b}{a+b} = \dfrac{c-d}{c+d}$.

If $\dfrac{x+y}{2a+b} = \dfrac{y+z}{2b+c} = \dfrac{z+x}{2c+a}$, prove that, in general,

$$\frac{x+y+z}{a+b+c} = \frac{(b+c)x+(c+a)y+(a+b)z}{2(ab+bc+ca)}.$$

6. Draw accurately, using the same axes and marking the scales, the graphs of x^2 and $x+1{\cdot}5$ from $x = -2$ to $x=2$.

Thence solve graphically the equation

$$2x^2-2x-3 = 0.$$

Second Paper. October, 1906.

1. Find the following products, grouping terms of the same type together:—

(i) $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$;

(ii) $(a+b+c)^2(a^2+b^2+c^2-bc-ca-ab)$.

2. Solve the following equations:

(i) $4(x^3+1)-13(x+1)^2=0$;

(ii) $\dfrac{x^2+x+1}{x^2-x-2}+\dfrac{x^2-x+1}{x^2-3x+2}=2$.

3. Find the quadratic function of x which has zero values when $x=3+\sqrt{2}$ and $x=3-\sqrt{2}$, and whose value is 10 when $x=5$.

4. If $12:x=x:y=y:z=z:18$, calculate the value of x to two places of decimals: and show that

$$\frac{x^4+y^4+z^4}{x^2+y^2+z^2}=x^2-y^2+z^2.$$

5. What are the main types of simultaneous equations involving two unknowns that can be solved by elementary methods? Ch. XIX, 4

Solve the simultaneous equations: §224.

$$2y-3x=2,\quad x^2+2xy=20.$$

The perimeter of a rectangle is 302 yards and its area is 5460 square yards; find the diameter of the circumscribing circle.

Mathematics. Lower Standard. April, 1907.

1-5. Arithmetic.

6. Find the factors of the following expressions:

(i) $12x^2+19ax-18a^2$;

(ii) $(x-1)(2x^2+5x+3)+(x+1)(2x^2-3x+1)$.

For what values of x does the expression (i) vanish?

7. (i) Express as a single fraction in its lowest terms:

$$\frac{1}{x+3}-\frac{2(2-x)}{3-x}-\frac{x-15}{x^2-9}.$$

(ii) Solve the equation:

$$\frac{x}{x-a} - \frac{x+a+b}{x+a} - \frac{a-b}{x} = 0.$$

8. For what values of A, B and C is the equation

$$\frac{8x+1}{x(4x^2-1)} = \frac{A}{2x-1} + \frac{B}{x} + \frac{C}{2x+1}$$

an identity?

9. Solve the equations:

(i) $(x^2+3x+2)^2 = 6(x^2+3x+2)$;

(ii) $x+6y-(3x-3y+5) = y+2x-2(2y+3x+3) = \frac{3}{4}y-x$.

10. Show that the square of the sum of any four numbers is equal to the sum of their squares together with twice the product of every pair.

11. A line 12 inches long is divided into two parts such that the square on one part is equal to three times the rectangle contained by the whole line and the other part; find to the nearest hundredth of an inch the lengths of the two segments.

MATHEMATICS. INTERMEDIATE AND HIGHER.

FIRST PAPER. MARCH, 1907.

1. Arithmetic.

2. State a rule for calculating rapidly the coefficients in the expansion of $(1+x)^n$ when n is not very large.

Find the coefficient of x^6 in

$$(x+1)^3(x-1)^5+(x+1)^5(x-1)^3.$$

3. Determine p in order that the system of equations

$$x+2y-3z = 0, \quad x+y-z = 0, \quad x+py+2z = 0$$

may admit of solutions other than $x = 0$, $y = 0$, $z = 0$.

Solve the system

$$x+2y-3z = 0, \quad x+y-z = 0, \quad x^2+y^2+z^2-x-y-z = 0.$$

4. Find the G. C. M. of

$$x^4-x^3+2x^2+x+3 \quad \text{and} \quad x^4-3x^3+6x^2-5x+3.$$

5. Two integers consist each of the same two digits, but in different order. If the sum of the two be 165 and their product 6786, find them.

Second Paper. April, 1907.

1. Solve the quadratic $x^2+21{\cdot}2x+39{\cdot}1=0$, giving each root correct to two places of decimals.

2. State the condition that x^2+px+q, where p and q are rational numbers, may be decomposable into two factors whose coefficients are rational numbers.

Factorize completely $x^6-6x^4+11x^2-6$.

3. Express $\dfrac{x-a}{x+a}+\dfrac{x-b}{x+b}+\dfrac{x-c}{x+c}$ as a single fraction.

4. If $x=(y+a)/(y+b)$, and $y=(z+a)/(z+b)$, for what particular values of x is x equal to z?

Solve the system $x^2+y^2+x+y=0$, $x^2-y^2+x-2y=0$.

5. If $a:b=b:c=c:d$, and a, b, c, d be all real positive quantities, prove that

(i) $a+d>b+c$; (ii) $\dfrac{a^2+b^2}{c^2+d^2}=\dfrac{a^2}{b^2}$.

6. Determine the general shape of the graph of

$$y=\frac{x-1}{x-2}.$$

UNIVERSITIES OF MANCHESTER, LIVERPOOL, LEEDS AND SHEFFIELD JOINT MATRICULATION EXAMINATION.

September, 1906.

1. The statement that 8 kilometres is equal to 5 miles gives the length of a kilometre too great by about $\frac{1}{2}$ per cent. Find, as accurately as possible from this statement. the length of a metre in yards.

2. What is the object of rationalizing the denominator of an arithmetical fraction involving surds? Calculate correct to four decimal places:

$$\frac{3\sqrt{2}+\sqrt{6}}{3+\sqrt{3}}.$$

3. Factorize (1) $x^3-7x-18$; (2) x^3-2x^2-5x+6;

(3) $2axy+y^2-x^2-a^2y^2$; (4) a^4+4b^4.

4. Explain (1) by arithmetical verification, and (2) by a geometrical illustration, the identity

$$a-(b-c)=a-b+c.$$

Solve the equation

$$\tfrac{1}{10}(4x+1)-\tfrac{1}{5}(7-x)=39-\tfrac{1}{2}(274-12x).$$

5. Reduce

$$\frac{2+x}{x+5}-\frac{3-x}{x-4}+\frac{6-x}{x+2}.$$

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, then $\frac{a+b}{c+d}=\left(\frac{a+c}{b+d}\right)^2$.

6. Solve $x^2=7x+6$ correct to two decimal places.

If ax^2+bx+c has the values 4 when x is 3, 20 when x is 4, and 34 when x is -3, find a negative value of x which will make the expression equal to 0.

7. The area of a certain rectangle is equal to the area of a square whose side is three inches longer than one of the sides of the rectangle. If the shorter side of the rectangle be decreased by one inch and the longer side increased by two inches, the area is unaltered. Find the lengths of the sides of the rectangle.

8. A certain Arithmetical Progression has its first term a and its last term l; what is the average of all the terms? If there is an odd number of terms what is the value of the middle one? (Answer this, if you can, by general reasoning without quoting formulae.) A man begins by walking $1\frac{1}{2}$ miles in the day and each day for 4 weeks he walks one mile more than the day before; what is his average daily walk during that time and what distance did he walk on the last day?

9. A hare runs ten times as fast as a tortoise which it is pursuing. The hare at A is 100 yards behind the tortoise at B; when the

hare is at B the tortoise has advanced to C, and when the hare is at C the tortoise has advanced to D, and so on an endless number of times. Find by adding up the pieces AB, BC, CD, &c., how far the hare will have gone when it overtakes the tortoise. If you can find the answer in any other way, show that the results of both methods agree.

JULY, 1907.

1 and 2 are Arithmetic.

3. If a polynomial in x vanishes when $x=a$, show that it contains $x-a$ as a factor.

Show that

$$bc(b-c)+ca(c-a)+ab(a-b)+(b-c)(c-a)(a-b)=0.$$

4. Find the factors of:

(1) x^3+x^2y-x-y;

(2) $2x^2+15x-77$;

(3) $1+x^3(y-z^3)-x^3yz$.

5. Solve the equations:

(1) $a(2x-a)+b(2x-b)=2ab$;

(2) $2x-3y=4, \quad 3x=5+7y$;

(3) $\dfrac{3}{x+1}+\dfrac{5}{x+3}=\dfrac{12}{x+4}$.

6. Find two numbers, one of which is $\frac{3}{5}$ths of the other, such that the difference of their squares is 6237.

7. If four quantities are in proportion, show that the difference between the first and second is to the second as the difference between the third and fourth is to the fourth.

If $(w+x+y+z)(w-x-y+z)=(w-x+y-z)(w+x-y-z)$, show that w, x, y, and z are proportionals.

8. Find the square root of the ratio of two numbers whose arithmetic mean is twice as great as their geometric mean.

9. Find the sum of n terms of a G. P. whose first term is a and the common ratio r.

The sum of the first 6 terms of a real G. P. is 65 times the sum of the first 3 terms. Find the common ratio.

COLLEGE OF PRECEPTORS.

First Class Certificate Examination.

Christmas, 1907.

1. If $a^2-bc=x^2$, $b^2-ca=y^2$, $c^2-ab=z^2$, show that
$$(a+b+c)(x^3+y^3+z^3)=ax^3+by^3+cz^3.$$

2. Simplify:

(i) $\dfrac{2x^3-13x+18}{2x^3-7x^2-15x+27}$;

(ii) $\dfrac{2x+3y}{8x^3+12x^2y-18xy^2-27y^3}+\dfrac{2x-3y}{8x^3-12x^2y-18xy^2+27y^3}$;

(iii) Divide $x^3-(a+b+c)x^2+(bc+ca+ab)x-abc$ by $x-c$. Find the factors of the quotient.

3. Solve the equations:

(i) $\dfrac{x-2}{x-3}-\dfrac{x-4}{x-5}=\dfrac{x-6}{x-7}-\dfrac{x-8}{x-9}$;

(ii) $\left.\begin{aligned}\dfrac{1}{3x+2y}&=\dfrac{7}{3y-x}\\ \dfrac{3}{(x-2)(y-1)}&=\dfrac{4}{y(1-y)}-\dfrac{6}{y(x-2)}\end{aligned}\right\}$;

(iii) $\sqrt{4x^2+2x+7}=(12x^2+6x+21)-140$.

Either 4 (*a*). A number of two digits is such that, if the digits be transposed, the new number is less by 9 than the original number, and the sum of the squares of the digits is 113. Find the number.

Or 4 (*b*). Draw the graphs of $y=2x+7$ and $y=7-2x$.

Take the points in which these lines respectively cut the axis of x, and deduce the graphs of the lines drawn through these points parallel to the original lines.

5. Find the square of $3\sqrt{3}-4\sqrt{5}+5\sqrt{15}$; and the square root of $152-30\sqrt{15}$.

Simplify $\dfrac{24\sqrt{5}-5\sqrt{3}}{2\sqrt{5}-3\sqrt{3}}$.

6. Divide $a+4b$ by $a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{4}}+2b^{\frac{1}{2}}$.

Find the square root of $a+b+\dfrac{4}{a+b}-4$.

7. If $x+y:3a-b::y+z:3b-c::z+x:3c-a$, prove that

$$\frac{x+y+z}{a+b+c}=\frac{ax+by+cz}{a^2+b^2+c^2}.$$

8. (i) In an Arithmetical Progression of an even number of terms, the sum of the odd terms is 105, the sum of the even numbers is 119, and the excess of the last term over the first is 26. Find the progression.

(ii) What is that Geometrical Progression of which the first term is 12, and the limit of the sum of the terms is 8?

(iii) What is the present value of £a due n years hence, compound interest being reckoned at c per cent.?

Either 9 (a). Explain what is meant by Variation; and if $a \propto b$, and $c \propto d$, prove that $ac \propto bd$, and $\dfrac{a}{c} \propto \dfrac{b}{d}$.

If $y = t+v$ where $t \propto x$ and $v \propto \sqrt{x}$, find the relation between x and y when, x being given equal to 4, y equals 5; and, when x is given equal to 9, y equals 10.

A passenger in a railway carriage observes another train, moving on a parallel line in the opposite direction, to occupy 2 seconds in passing him, but, when moving in the same direction, to occupy 30 seconds in passing. Compare the rates of the two trains.

Or 9 (b). Graph the function $10+8{\cdot}4x-0{\cdot}6x^2$. Find its maximum value, and the values of x for which it vanishes.

Find as accurately as you can, the maximum or minimum value of (i) $(2x+3)(x-\frac{1}{2})$, (ii) $x(12-x)$; and state the value of x for which the function has its turning value.

[Any of the following questions A, B, C, D may be substituted for the same number of the questions 5, 6, 7, 8, 9, but will receive considerably lower marks.]

A. Simplify

$$\left(a^2-\frac{4a^2x-3ax^2+x^3}{a+\dfrac{1}{x^{-1}}}\right)\div\left(a^2+\frac{4a^2x+3ax^2+x^3}{a-\dfrac{1}{x^{-1}}}\right).$$

B. Find the L. C. M. of $(x+1)$, (x^2-1), (x^3-1), and (x^4-1).

C. A waterman went down a river and back again in 6 hours and found that he could row 9 miles an hour with the stream and 3 miles an hour against it. How far did he row down ?

D. Three numbers whose sum is 15 are in Arithmetical Progression, and if 1, 4, 9 be respectively added to them, the numbers will be in Geometrical Progression. Determine the numbers.

First Class Certificate. Midsummer, 1908.

1. Express $n^2(n+1)^2-6n(n+1)(2n+1)+36n(n+1)$ as a continued product; and find the simplest form of

$$6x^2+13xy+6y^2+y-x,$$

where $2x+3y=1$.

2. Simplify:

(i) $\dfrac{4}{x^2+3xy-10y^2}-\dfrac{1}{x^2-2xy-35y^2}+\dfrac{6}{x^2-9xy+14y^2}$;

(ii) $\dfrac{a(a+b)x^2+b^2xy-a(a-b)y^2}{a(a+b)y^2+(a^2+ab+b^2)xy-b(a+b)x^2}$.

3. Prove that, if

$xy=3(x-3)$ and $yz=3(y-3)$, then $xz=3(z-3)$.

4. Solve the equations:

(i) $\dfrac{2x-1}{4x-3}-\dfrac{6x-5}{14x+3}=\dfrac{1}{2}$;

(ii) $\left.\begin{aligned} b(x-3b)&=a(y+2b)\\ (a+2b)x-2(a+b)y&=4a^2+7ab+2b^2 \end{aligned}\right\}$.

Either 5 (*a*). If a dealer sold 1512 articles for £3. 9*s*. more than they cost, and at such a rate that the number sold for 1*s*. exceeded by unity one-third of the number of dozens bought for £1, what did he give for the articles ?

Or 5 (*b*). The highest and lowest marks awarded at an examination were 285 and 75 respectively. All the marks have to be reduced so that the highest and lowest marks may be 200 and 60 respectively. Draw the necessary graph, and from it determine what marks of the new system correspond to 95, 175, 215, and 255 of the old system.

Either 6 (*a*). Solve the equations:

(i) $\sqrt{15x-2}-\sqrt{5x-1}=\sqrt{20x-59}$;

(ii) $(x^2-6x+3)^2+6(x-3)^2=x^2+27$.

Or 6 (*b*). Carefully draw the graph of $y=2x^2-6x+7$, and from it find the minimum value of $2x^2-6x+7$ and the approximate roots of the equations:

(i) $2x^2-6x+3=0$; (ii) $2x^2-6x+1=0$;

(iii) $2x^2-6x-3=0$.

7. Simplify:

(i) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}\times\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$; (ii) $\dfrac{\sqrt{126+45\sqrt{3}}-4\sqrt{2}}{\sqrt{38+21\sqrt{3}}}$.

8. If a and b are unequal quantities and

$$ab(c^2+d^2)=b^2c^2+a^2d^2,$$

then the ratio of a to b is the duplicate ratio of c to d.

Given that a, b, c are three positive quantities, and that

$$\frac{b}{a+b}=\frac{a+c}{a+4b-c}=\frac{a+b+c}{3b+2c},$$

determine the ratios $a:b:c$.

9. Find the Geometrical Progression, such that the sum of the first 4 terms is $\frac{2}{3}\frac{5}{4}$ and the sum to infinity $1\frac{1}{2}$.

If a, b, c are in Arithmetical Progression, and b, c, d in Harmonical Progression, then

$$ab(c^2+d^2)=cd(a^2+b^2).$$

[Candidates may substitute any one of the following questions, A, B, C, D, for the same number of the questions 6, 7, 8, 9; but the questions A, B, C, D will receive considerably fewer marks.]

A. If $15x^3-4x^2+ax+8$ is exactly divisible by $3x-2$, what is the value of a?

B. Solve the equation $\dfrac{x+b}{2a+3b}-\dfrac{3x-7b}{4(3a+2b)}=1$.

C. Simplify $\left(\dfrac{a^{-\frac{3}{4}}b^{-\frac{1}{4}}}{a^2b^{\frac{1}{2}}}\right)^{\frac{4}{5}}\div\left(\dfrac{a^{-\frac{3}{4}}b^{\frac{1}{2}}}{a^{\frac{1}{4}}b^{\frac{3}{2}}}\right)^6$.

D. Find the square root of

$$(a^2+b^2)(a^2+c^2)-2a(b+c)(a^2+bc)+4a^2bc.$$

OXFORD LOCAL EXAMINATIONS.

JUNIOR CERTIFICATE, 1907.

1. Find the H.C.F. of x^4+3x^3+4 and x^3+x^2-2.

2. Find the square root of $4x^6+4x^5-3x^4-10x^3-3x^2+4x+4$.

3. Simplify $\left\{1-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right\}\left\{1+\frac{2}{x-2}+\frac{1}{(x-2)^2}\right\}$.

4. Show that

$$(a^2-bc)(b^2-ca)(c^2-ab)=abc(a^3+b^3+c^3)-(b^3c^3+c^3a^3+a^3b^3),$$

and verify the result when $a=2$, $b=-3$, $c=-1$.

5. Solve the equations:

(1) $\frac{2}{x}+\frac{3}{x+2}=\frac{5}{x+1}$.

(2) $\frac{x-a}{b}+\frac{x-b}{a}=2$.

(3) $6x^2+13x+6=0$.

6. Show by drawing graphs that the values of x and y which satisfy the equations

$$3x+y=2,\ 7x+4y=3,$$

satisfy also the equations

$$2x+5y+3=0,\ 5x+2y-3=0.$$

7. (1) Find the cube of $x^{\frac{1}{2}}+y^{\frac{1}{2}}$.

(2) If $\frac{a+x}{b+x}=\frac{b^n}{a^n}$, and $\frac{a+y}{b+y}=\frac{a^{n+1}}{b^{n+1}}$, show that $xy=ab$.

8. At an election contest between two candidates the total number of votes recorded was 8600. Had one-third of the successful candidate's supporters stayed away and half the unsuccessful candidate's, the successful candidate's majority would have been reduced by 200. How many votes did each actually receive?

HIGHER ALGEBRA.

1. If $\frac{x}{a(b+c-a)}=\frac{y}{b(c+a-b)}=\frac{z}{c(a+b-c)}$, prove that

$$(y+z-x)(b+c-a)=(z+x-y)(c+a-b)$$
$$=(x+y-z)(a+b-c).$$

2. An arithmetic and a geometric progression have their first two terms the same, and the ratio of the sum of the first three terms of the arithmetic progression to the sum of the first three terms of the geometric progression is $\frac{9}{7}$. Prove that the ratio of the sum of the first four terms of the arithmetic progression to the sum of the first four terms of the geometric progression is either $\frac{3}{2}$ or $\frac{8}{15}$.

3. Write out in full the permutations of the letters a, b, c, d two at a time, and the combinations of the letters a, b, c, d, e three at a time.

In how many ways can 9 balls be put into two bags, neither of which holds more than 6?

4. Assuming the binomial theorem for a positive integral index, prove it for a positive fractional index.

Show that the coefficient of x^n in the expansion of

$$(1-x)^{-3} \text{ is } \tfrac{1}{2}(n+1)(n+2).$$

SENIOR. 1906.

1. Find the remainder when $x^4-3x^3-5x^2+20x-8$ is divided by $x-3$.

2. Find the square root of

$$\frac{x^4}{4}+2x^3+3x^2-5x-3+\frac{2}{x}+\frac{1}{x^2}.$$

3. (i) Divide $x^{\frac{3}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}-2y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

(ii) Find the L.C.M. of

$$x^3-2x^2+x-2,\ x^3-3x^2+x-3, \text{ and } x^3-4x^2+x+6.$$

4. Show that

$$\frac{a(b+c)}{(a-b)(a-c)}+\frac{b(c+a)}{(b-c)(b-a)}+\frac{c(a+b)}{(c-a)(c-b)}+1=0.$$

5. Solve the equations:

(i) $a(x+y)=b(x-y)$, $\frac{a}{x}-\frac{b}{y}=\frac{a^2+b^2}{a^2-b^2}$;

(ii) $2x^2-5x+1=0$, finding each root correct to three places of decimals.

6. Show that, if a is a root of the equation

$$ax^2+bx+c=0,$$

then $x-a$ is a factor of the expression

$$ax^2+bx+c.$$

7. The sum of the squares of two quantities varies as the sum of the quantities, and, when the quantities are equal but not zero, each of them is equal to 2. Find the values of one when the other is equal to $\frac{1}{2}$.

8. Show by examples the difference between permutations and combinations.

In how many ways can a party of 16 people be conveyed in two vehicles, one of which will not hold more than 8 people, and the other not more than 10?

9. Find (i) the coefficient of x^6 in the expansion of $(1+x)^{10}$, and (ii) the coefficient of x^4 in the expansion of $(1+x+2x^2)^4$.

Higher Algebra. 1906.

10. Prove that, if x is numerically less than unity,

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots.$$

11. Sum the series

$$1^3+2^3+3^3+\ldots+n^3.$$

12. Show that, if $a_1, a_2, \ldots a_n$ are any positive quantities,

$$\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n \not< a_1a_2\ldots a_n.$$

Find the greatest value of the expression

$$(2a-3x)^2(3a+2x)^3$$

for values of x lying between $\frac{-3a}{2}$ and $\frac{2a}{3}$.

13. Draw the graph of the function 2^x.

Show that an equation of the form $2^x=ax$, where a is a real positive quantity, has either two real roots or none.

14. Evaluate the determinant

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ a, & b, & c, & d \\ a^2, & b^2, & c^2, & d^2 \\ a^4, & b^4, & c^4, & d^4 \end{vmatrix}.$$

SENIOR. 1907.

1. Find the H. C. F. of

$$x^4-x^3-4x-4 \text{ and } 2x^4+3x^3+4x^2+x-2.$$

2. (1) Find the square root of

$$x^3+2x^{\frac{5}{2}}-2x^{\frac{3}{2}}+x-2x^{\frac{1}{2}}+1.$$

(2) Simplify

$$\frac{1}{1-x}+\frac{3}{1+x}+\frac{2}{(1+x)^2}+\frac{2(1-x)}{1+x^2}.$$

3. (1) Find x and y from the equations

$$\frac{4x}{a}-\frac{y}{3b}=c, \quad \frac{7x}{a}+\frac{y}{2b}=5c.$$

(2) Find x from the equation

$$\frac{1}{x+1}+\frac{1}{x+2}-\frac{3}{5x+2}=0.$$

4. If $a+c=2b$, show that $a^2-2b^2+c^2=(a-b)(a-c)$.

5. Solve the equation $ax^2+bx+c=0$.

Find the roots of the equation $x^2-7x-1=0$, correct to two places of decimals.

6. (1) Find the sum to n terms of an arithmetic progression, of which the first term is a and the common difference d.

(2) The sum of the first eight terms of a geometric progression is seventeen times the sum of the first four terms. Find the value of the common ratio.

7. The product of two quantities varies as the excess of their sum over 3, and, when one of the quantities is equal to 2, so is the other. Find what other value the quantities may have simultaneously.

8. Show, without using any formulae, that the number of combinations of $m+n$ things taken m together is the same as the number taken n together.

A party of four people is to be chosen from nine, among whom are a Mr. and Mrs. A. and a Mr. and Mrs. B. Mr. and Mrs. A., if invited at all, must be invited together; similarly, Mr. and Mrs. B., if invited at all, must be invited together. In how many ways can the party be chosen?

9. Write out in full the expansion of $(a-x)^5$.

Find the coefficient of x^3 in the expansion of $(1+ax+x^2)^n$.

HIGHER ALGEBRA, 1907.

1. Show that the number of ways in which a council of $2n+1$ members, representing three parties, may be constituted so that no party has an absolute majority is the coefficient of x^{2n+1} in

$$(x+x^2+x^3+\ldots+x^n)^3.$$

Show also that this coefficient is equal to $\frac{1}{2}n(n+1)$.

2. Resolve into partial fractions

$$\frac{1}{(1-x)^2(1+x^2)}.$$

3. Convert $\sqrt{33}$ into a recurring continued fraction.

4. Show that $1+\underline{|n-1}$ is divisible by n, if n is a prime number.

Show conversely that, if $1+\underline{|n-1}$ is divisible by n, then n must be a prime number.

5. Evaluate the determinant:

$$\begin{vmatrix} x, & x+1, & x+2 \\ x+1, & x+2, & x+3 \\ x+2, & x+3, & x+4 \end{vmatrix}.$$

CAMBRIDGE LOCAL EXAMINATIONS.

JUNIOR. DECEMBER, 1907.

[Candidates can pass in this subject by doing sufficiently well in Part I of the paper.]

PART I.

1. Find the value of $(x-y)^2+2(z+x)^2-3(x+y)(y-z)$ when $x=1$, $y=-2$, $z=3$.

2. Multiply $x^2-4x+16$ by $x^4+4x^2-5x-20$.
Divide $x^4-5x+156$ by $x^2-5x+12$.

3. Find all the simplest factors of
(i) x^3+2x^2-35x, (ii) $(a^2+1)y^2-y^4-a^2$.

4. Simplify
$$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-z)(y-x)}+\frac{(x+y)z}{(z-x)(z-y)}.$$

5. Solve
(i) $\frac{x+a}{3x-1}+\frac{a-x}{1+3x}=\frac{3a+1}{9x^2-1}$;

(ii) $4x-9y=8$, $\frac{y-x}{7}=x+y$;

(iii) $3x^2+16x=299$.

6. Plot the graphs, from $x=-1$ to $x=+4$, of the equations
$$y=\tfrac{1}{2}(x-2)^2, \text{ and } y=\frac{10-x}{6};$$
and find, from the diagram, the values of x at the points where they intersect.

7. Two boys, A and B, have each a certain number of nuts, B's being the greater number. If A were to give B 18 nuts, B would have twice as many as A; but if B were to give A three times as many nuts as the difference between their original numbers, then A would have twice as many as B. How many nuts has each?

PART II.

8. Prove that a greater inequality is increased by subtracting the same positive quantity from each of its terms.

What quantity must be subtracted from each of the terms of the ratio of 113 to 71, to produce the ratio 3 to 1?

9. Find the sum of n terms of an Arithmetical Progression whose first term is a and common difference b.

Show that the sum of the first m terms of this Progression will be equal to the sum of the first n terms if

$$b = -\frac{2}{m+n-1}a.$$

10. Find $P_{n,r}$, the number of permutations of n things taken r at a time; and prove that

$$P_{n+1,r+1} - P_{n,r+1} = (r+1)P_{n,r}.$$

From 10 red and 7 black counters how many different rows can be made with 6 counters in each row?

11. Prove the Binomial Theorem for a positive integral exponent.

If $(1+x)^3 = 1 + a_1x + a_2x^2 + \ldots$

and $(1+x)^5 = 1 + b_1x + b_2x^2 + \ldots,$

find the sum of $1 + a_1b_1 + a_2b_2 + a_3b_3.$

12. Prove that $a^m a^n = a^{m+n}$.

Which is greater, $10^{\frac{1}{2}}$ or $2^{\frac{3}{2}}$? Justify your answer.

13. State what is meant by the expression $\log_2 32$, and find its numerical value.

Prove that $\log_e a^{\frac{1}{m}} = \frac{1}{m}\log_e a$, when m is a positive integer.

Given $\log_{10}2 = .301$, $\log_{10}3 = .477$, $\log_{10}7 = .845$; write down a table of the logarithms of the first ten integers; and show from your table that

$$5 = 3^{1.465} \text{ approximately.}$$

SENIOR. JULY, 1907.

[N.B. Attention is called to the alternative questions A, B, C, D at the end of the paper.]

1. Simplify:

(i) $\left(\frac{x+a}{x-a}+\frac{x-a}{x+a}\right) \div \left(\frac{x+a}{x-a}-\frac{x-a}{x+a}\right)$;

(ii) $\frac{3a^2+8ab-3b^2}{9a^2-b^2} \times \frac{6a^2-7ab-3b^2}{2a^2+5ab-12b^2}$.

2. Find the H. C. F. of

$x^4-4a^2x^2+3a^3x+6a^4$ and $ax^4+3a^2x^3+a^3x^2-2a^4x$.

Simplify $\frac{a}{x-a}-\frac{a}{x+a}-\frac{2a^2}{x^2+a^2}-\frac{4a^4}{x^4+a^4}$.

3. Solve the equations:

(i) $\frac{1}{3x-3}+\frac{1}{4x+4}-\frac{7}{12x}=0$;

(ii) $x^2-a(3x-2a)+b(x-2a)=0$;

(iii) $2x^2+y^2=43$,
$x^2+xy=24$.

§168.

4. What is the price of gold per ounce, if a rise of 3*d.* reduces by 5 the number of ounces that can be bought for £5757 ?

5. Plot the graphs $y=\frac{1}{x}+1$, $y=x^2+2x$, between the values $x=-3$ and $x=2$, using the same axis for both graphs, and taking ten of the small divisions of the squared paper as the unit.

Hence obtain approximate solutions of the equation

$$x^3+2x^2-x-1=0.$$

6. Find the n^{th} term of an Arithmetical Progression whose first two terms are a and b.

If, for all values of n, the sum of n terms of an Arithmetical Progression is n^2+6n, find the n^{th} term.

7. Prove that $\log_a m^\kappa = \kappa \log_a m$.

Write down the logarithms of (i) 64 to the base 2, (ii) 2 to the base 64.

Evaluate the seventh root of ·8912.

8. Find for what value of r the number of combinations of n unlike things taken r together is greatest.

If two out of n things are alike and the rest are unlike, prove that the number of combinations of them taken r together is (with ordinary notation)

$$^{n}C_{r} - {}^{n-2}C_{r-2}.$$

9. Write down the coefficient of x^r in the expansion of $(1-x)^{-\frac{1}{4}}$ by the Binomial Theorem.

If x is so small that its cube and higher powers may be neglected, prove that

$$\frac{(1-4x)^{\frac{1}{2}} + (1-3x)^{\frac{1}{2}}}{(1-2x)^{\frac{1}{2}}} = 2 + \frac{3x^2}{4}.$$

10. Find the conditions that the expression ax^2+bx+c may be positive for all real values of x.

Prove that, for real values of x, $\dfrac{x^2-3x+1}{2x^2-3x+2}$ must lie between -1 and $\frac{5}{7}$.

11. Show how to continue the series

$$1+3x+10x^2+36x^3+\ldots$$

as a recurring series of the second order, and find the n^{th} term.

12. Find the sum of the cubes of the first n natural numbers, and prove by induction or otherwise that

$$s_7 + s_5 = 2(s_3)^2,$$

where $$s_r \equiv 1^r + 2^r + 3^r + \ldots + n^r.$$

[N.B. Any of the following questions may be taken instead of any of the questions 9, 10, 11, 12, but considerably lower marks will be assigned to them.]

A. Resolve into factors:

(i) $(b-c)^3 + (c-a)^3 + (a-b)^3$;

(ii) $x^4 + x^2y^2 + y^4$.

B. Solve the equations:

(i) $\dfrac{7}{x} - \dfrac{4}{y} = 3, \quad \dfrac{5}{x} + \dfrac{2}{y} = 6$;

(ii) $(x-5)(2-x) + \dfrac{10x}{7} = 0.$

C. The sum of £40. 19*s*. 9*d*. is divided between 60 men and women so that each man receives as many shillings as there are men and each woman receives as many pence as there are women. How many men are there?

D. If $\frac{x}{y}=\frac{a-b}{a+b}$, prove that $\frac{y-x}{x+y}=\frac{b}{a}$.

If $\frac{x+2y}{r+2q}=\frac{y+2z}{p+2r}=\frac{z+2x}{q+2p}$, prove that

$$x:y:z=2p+2q-r:2q+2r-p:2r+2p-q.$$

Senior. December, 1907.

1. Resolve into factors:

(i) $(a+2b)^2-(2a+b)^2$;

(ii) a^4-8a.

Simplify $\left(\frac{a}{b}-\frac{b}{a}\right)\times\left(\frac{a}{b}-1+\frac{b}{a}\right)\div\left(\frac{a}{b}-2+\frac{b}{a}\right)$.

2. Reduce to lowest terms $\frac{9a^4-3a^2+1}{9a^3+3a^2+3a-1}$.

Simplify $\frac{3}{2-2x}-\frac{4}{3+3x}+\frac{5}{4x^2-4}$.

3. Solve the equations:

(i) $\frac{6x+4a}{2x+a}-\frac{2x+a}{x+a}=1$;

(ii) $\frac{x-1}{x+1}-\frac{x-2}{x+2}=\frac{x-3}{x+3}-\frac{x-4}{x+4}$;

(iii) $x^2+4y^2=17$,
$3x-2y=-1$.

4. A number of similar articles were bought for £4. 10*s*., and the purchaser, by selling them for 6*d*. each more than the cost price, received for all but 6 as much as all had cost. How many articles were there?

5. Plot the graphs $y = \pm\sqrt{x}$, $y = \frac{x^2}{4} - 2$ between the values $x = -2$ and $x = 4$, using the same axes for both graphs and taking ten of the small divisions of the squared paper as the unit.

Hence obtain the approximate solutions of the simultaneous equations $$x^2 - 4y = 8, \quad x - y^2 = 0.$$

6. Prove that if m and n are positive integers, $$a^m \times a^n = a^{m+n}.$$

Simplify $[a^{\frac{1}{2}}\{a^{-\frac{1}{2}}b^{-\frac{1}{2}}(a^2b^2)^{-\frac{1}{2}}\}^{\frac{1}{2}}]^6$.

Evaluate with the help of logarithm tables $$(123{\cdot}4)^{\frac{1}{2}} \times (\cdot 007)^{\frac{1}{3}}.$$

7. Prove the formula for finding the sum of n terms of a Geometrical Progression.

The fifth term of a Geometrical Progression is 64 and the eleventh term is 729. Find the thirteenth term.

8. Find the number of permutations of n unlike things r at a time.

In how many ways can 10 persons be seated in a railway carriage, 5 on each side, if 3 of them will not sit back to the engine?

9. Find the term independent of x in the expansion of $$\left(2x^2 - \frac{a}{2x^3}\right)^{10}.$$

Obtain the first four terms in the expansion of $$\frac{1}{(2-x)(2-3x)^{\frac{1}{2}}}.$$

10. Resolve into partial fractions $$\frac{x^2}{(x-1)^2(x-2)(x-3)},$$ and obtain the sum of the first n coefficients in the expansion in powers of x of the expression $$\frac{x^2}{(x-1)(x-2)(x-3)}.$$

11. Prove that the arithmetic mean of any number of positive quantities is greater than their geometric mean.

Prove that $\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{2} < \left(\frac{a+b}{2}\right)^{\frac{1}{2}}$.

12. Prove the law of formation of successive convergents of the continued fraction

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots.$$

Prove that if $\frac{p_n}{q_n}$ is the n^{th} convergent of the recurring continued fraction $\frac{2}{1} + \frac{2}{1} + \frac{2}{1} + \dots$,

then $p_n + p_{n-1} = 2^n$.

[N. B. Any of the following questions may be taken instead of any of the questions 9, 10, 11, 12, but considerably lower marks will be assigned to them.]

A. If $a^3 + b^3 + c^3 = 3abc$, prove that either

$$a + b + c = 0 \quad \text{or} \quad a = b = c.$$

B. Solve the equations:

(i) $\frac{2x-y}{3} = \frac{2x+y-3}{6}$, $\frac{3x}{4} = \frac{y}{3} + 2\frac{1}{3}$;

(ii) $\frac{x-2}{2x-5} - \frac{x-3}{3x-7} = \frac{7}{15}$.

C. Find the square root of

$$6a^3b^2 - 4a^2b^3 + b^4 - 12a^5b + 9a^6 + 4a^4b^2.$$

D. Find a number of two digits which is equal to seven times the sum of its digits, and exceeds the sum of the squares of its digits by 4.

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD.

EXAMINATION FOR SCHOOL CERTIFICATES. JULY, 1906.

PART I.

[This has been given on page 320, Part I.]

PART II.

7. Solve the equations:

$$5y - x = 8,$$
$$9y^2 - x^2 = 48a - 16a^2.$$

8. Prove that the roots of the equation

$$x^2+px+q=0$$

will be unreal if $4q$ is greater than p^2.

Find, in terms of p and q, the sum of the squares and the sum of the fourth powers of the roots of the above equation.

9. Define the logarithm of a number to base 10, and explain why tables need only contain the logarithms of numbers between 1 and 10.

Find by the tables the values of:

(1) The number whose logarithm is 3·5;

(2) $(4827)^{\frac{1}{3}}$.

10. Draw the graph of 2^x for values of x between -3 and $+3$ inclusive, and find approximately from your figure the value of x which satisfies the equation $2^x=5$.

11. If $a:b::b:c::c:d$, show that

(1) $\dfrac{la^3+mb^3}{lb^3+mc^3}=\dfrac{a}{d}$;

(2) $b^2c+ad^2=abd+c^3$.

12. Sum to 15 terms the series:

(1) $-3+5+13+21+\ldots$;

(2) $(1+2)+(2+2^2)+(3+2^3)+(4+2^4)+\ldots$.

Lower Certificates. July, 1906.

1. Simplify

$$2(a-2b)^3-b(a+3b)^2+4b^2(a+5b)+5b^3.$$

2. Divide

$$6a^4+25a^3x+17a^2x^2-28ax^3-20x^4 \text{ by } 3a^2-ax-2x^2.$$

3. Resolve into factors:

(i) $6x^2+7x-5$;

(ii) $9a^2-b^2+4bc-4c^2$.

4. Simplify:

(i) $\dfrac{a-b}{a(a+b)} - \dfrac{a+b}{b(b-a)} - \dfrac{3a-b}{a^2-b^2}$;

(ii) $\dfrac{\dfrac{x}{y} - \dfrac{x-y}{x+y}}{\dfrac{y}{x} - \dfrac{y+x}{y-x}} \div \dfrac{\dfrac{x}{x+y} - 1}{\dfrac{x}{x-y} - 1}$.

5. Solve the equations:

(i) $\dfrac{2x}{5} - \dfrac{x-3}{3} = 3(x-\frac{2}{5})$;

(ii) $\dfrac{a}{x+b} + \dfrac{b-a}{x-b} = \dfrac{b}{x}$.

6. Find the Highest Common Factor of

$$4x^3+8x^2-9x-18 \text{ and } 4x^4+4x^3-5x^2-9x-9.$$

7. A man, who worked for 16 days, earned £5. 9*s*. in the last 5 days and thereby made the average of his daily earnings for the whole time greater by 4*s*. than the average for the first 11 days. Find his average daily earnings.

8. Solve the equations:

(i) $\frac{1}{13}(2x+7y) = \frac{1}{2}(3x-5y) = 4$;

(ii) $(5-2x)(x-1) = x(x+8)-7$.

9. If the four quantities a, b, c, d are in proportion, prove that

$$a+b : a-b :: c+d : c-d.$$

If $x : y :: -1 : 2$, find the value of

$$2x+3y : 5x+7y.$$

10. Find the sum of n terms of an arithmetical progression whose first and n^{th} terms are a and b respectively.

The third and seventh terms of a geometrical progression are $\frac{27}{8}$ and $\frac{8}{3}$ respectively. Find the sum of twelve terms.

11. Prove that $\log a^b = b \log a$.

Find approximately the value of $(12{\cdot}7)^{\frac{1}{3}} \div (.563)^{\frac{1}{11}}$.

12. Draw the graph of $y = \frac{1}{2}(1-x)(2x+7)$ for values of x between -4 and 3. Find, from the graph or otherwise, the greatest value of y.

Higher Certificate. July, 1906.

Part I.

[This is given on page 321, part I.]

Part II.

7. Solve the equations:

$$(1)\quad \begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13, \\ \dfrac{1}{x} + \dfrac{1}{y} = 5. \end{cases}$$

$$(2)\quad \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} = 3.$$

8. *A* has an old motor-car and travels 16 miles an hour, stopping 6 minutes at the end of each hour to overhaul his car. *B* on a new car travels continuously at 32 miles per hour, and starts 1 hour after *A*. Find, graphically or otherwise, when *B* overtakes *A*.

9. Prove that $a^m \times a^n = a^{m+n}$, when m and n are positive integers.

How do you obtain the identity $a^{-\frac{1}{2}} = \dfrac{1}{\sqrt{a}}$?

Simplify $\sqrt{a^{-\frac{1}{2}}b^3c^{\frac{3}{2}}} \div \sqrt[3]{a^{\frac{1}{2}}b^4c^{-1}}$.

10. If $a : b = c : d$, show that

$$a+b : a-b = c+d : c-d.$$

If $b+c$ is a mean proportional between $a+b$ and $c+a$, show that

$$b+c : c+a = c-a : a-b.$$

11. Prove that $\log_{10}a + \log_{10}b = \log_{10}ab$.

Find the number whose logarithm is $\bar{2}\cdot7049$, and the value of $\sqrt[5]{\cdot3928}$.

If s is a constant multiple of $t^{3}/_{2}$, and if $s = 31{\cdot}6$ when $t = 2{\cdot}93$, find t when $s = 8{\cdot}4$.

12. Prove the formula which gives the sum of n terms of a geometrical progression whose first term is a and the common ratio r.

Express as a decimal the sum of 8 terms of the series

$$4+\tfrac{4}{5}+\tfrac{4}{25}+\ldots.$$

Sum the series $-7-3+1+5+\ldots$ to 22 terms.

Higher Certificate. July, 1906.

Distinction Paper.

1 If α, β are the roots of $ax^2+bx+c=0$, and α', β' the roots of $a'x^2+b'x+c'=0$, show that

$$(c\alpha'-c'a)^2-(ab'-a'b)(bc'-b'c)=a^2a'^2(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha')(\beta-\beta').$$

2. (1) Solve the equation

$$[bc-x(b+c)]^3-4bcx[bc-x(b+c)](x-b-c)+8x^3b^2c^2=0,$$

having given that $\dfrac{bc}{b-c}$ is one of its roots.

(2) Solve the simultaneous equations:

$$\frac{x^3}{y}+xy=9,$$

$$\frac{y^3}{x}+xy=16.$$

3. Find the number of combinations of n things taken r at a time.

If this number be denoted by ${}_nC_r$, show that

$$1-{}_nC_1\frac{x-1}{x+1}+{}_nC_2\frac{x-2}{x+2}-\ldots+(-)^n\frac{x-n}{x+n}=\frac{2(n!)}{(x+1)(x+2)\ldots(x+n)}.$$

4. If $f(x)$ denote the infinite series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,$$

show that $f(x)\times f(y)=f(x+y)$.

Sum the series

$$1-\frac{2^3}{1!}+\frac{3^3}{2!}-\frac{4^3}{3!}+\ldots.$$

5. If x can be expressed as a series of powers of y, convergent when $y < 1$, by the formula

$$x = y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots,$$

express y as a series of powers of x, and find the values of x for which the series is convergent.

6. Explain the method of partial fractions and use it to obtain A, B, C in the equality

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.$$

Hence, or otherwise, show that

$$\frac{1}{(x-a)(x-b)(x-c)} - \frac{a+b+c-abc}{(1+a^2)(1+b^2)(1+c^2)}$$

$$= \frac{1+ax}{(1+a^2)(x-a)(b-a)(c-a)} + 2 \text{ similar terms.}$$

7. Show that the series

$$S \equiv u_0 + u_1 + u_2 + \ldots + u_n + \ldots,$$

all of whose terms are positive, is convergent, if

$$\operatorname{Lt}_{n=\infty} \frac{u_{n+1}}{u_n} < 1;$$

and divergent; if the same limit is > 1.

If, when n is very large, $\frac{u_{n+1}}{u_n}$ can be expanded in the convergent series

$$a_0 - \frac{a_1}{n} + \frac{a_2}{n^2} - \ldots,$$

show that the series S is only convergent when $a_0 < 1$, or when $a_0 = 1$ and $a_1 > 1$.

Discuss the convergency of the series

$$1 + \frac{ab}{cd}x + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)}x^2 + \ldots,$$

a, b, c, d, and x being real quantities.

8. If p_n/q_n be the n^{th} convergent to the continued fraction

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots,$$

show that $$p_n = a_n p_{n-1}+p_{n-2}.$$

If the a's are positive integers, show that

$$q_{2n+1} > 2q_{2n-1}.$$

9. Express as a determinant of the third order the product of two given determinants each of the third order.

Show that the determinant

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

is a perfect square.

10. Show how to obtain an algebraic solution of the cubic

$$x^3+3Hx+G=0.$$

If α, β, γ are the roots of the cubic

$$x^3+px^2+qx+r=0,$$

find the equation whose roots are

$$\alpha(\beta+\gamma),\ \beta(\gamma+\alpha),\ \gamma(\alpha+\beta).$$

11. Draw a rough graph of the function

$$f(x) \equiv x^3-3x+1.$$

Find, by Horner's process, the value of the negative root of $f(x)=0$ to 4 places of decimals.

LEAVING CERTIFICATE EXAMINATION (SCOTLAND).

Higher Grade, 1906.

1. If p pounds accumulate for n years at r per cent. compound interest, find an expression for the amount at the end of that time.

As an example, find to the nearest hundred pounds the amount due after £5000 has accumulated for 25 years at $4\frac{1}{2}$ per cent.

2. Simplify the following expressions:

(1) $\dfrac{1}{x+1}+\dfrac{2}{x^2+3}+\dfrac{8}{(x^2-1)(x^2+3)}$;

(2) $x^3+y^3+z^3+3(x+y+z)(yz+zx+xy)-(x+y+z)^3$.

3. If any integral function of x is divided by $x-a$, prove that the remainder is the expression formed by substituting a for x in the function.

If 3 is one root of the equation $x^3-49x+a=0$, find the other roots.

4. If N, N' are numerical quantities of the form $a+b\sqrt{2}$, $a'+b'\sqrt{2}$, where a, b, a', b' are integers and $N=N'$, prove that $a=a'$, $b=b'$.

Simplify:

(1) $\sqrt{2}(1+\sqrt{3})\sqrt{(2-\sqrt{3})}$;

(2) $2(2+\sqrt{3})(\sqrt{6}-\sqrt{2})\sqrt{(2-\sqrt{3})}$.

5. A boiler consists of a cylinder with hemispherical ends. If the length of the cylinder be 10 feet, and the diameter of its cross-section 5 feet, find its volume in cubic feet to the nearest foot, taking π as $\frac{22}{7}$.

6. Simplify and resolve into factors

$$(3x^2+1)(3x^2+3x+1)(3x^2-3x+1)$$
$$-(x^2+3)(x^2+3x+3)(x^2-3x+3).$$

7. Find all the real roots of the equations:

(1) $12\left(x+\dfrac{1}{x}\right)^2-56\left(x+\dfrac{1}{x}\right)+65=0$;

(2) $x^3-y^3=91$, $x^2(x-y)=36$.

8. Three numbers are in harmonic progression. If the sum of the numbers be 37, and the sum of the reciprocals be $\frac{1}{4}$, find the numbers.

9. Draw in one figure and with the same scale and axes the graphs of (1) $y = 3 - \frac{x}{2}$, (2) $y = \frac{5}{x}$; from $x = 1$ to $x = 6$, taking one inch as the unit.

Either directly from these graphs, or by means of their tables of values, form the graph of

$$(3) \quad y = \frac{5}{x} - 3 + \frac{x}{2},$$

and find its minimum or turning-point as nearly as you can from your figure.

Honours Grade, 1907.

1. Prove that the arithmetic mean of any number of positive quantities is greater than the geometric mean.

Find the least value of

$$15x + 10y + 6z$$

when $xyz = 30$; all the symbols denoting positive numbers.

2. If $\phi(x)$ be any rational integral function of x, prove that it is always possible to determine A and B so that $\phi(x) + Ax + B$ shall be divisible by $x^2 + c^2$.

Hence, or otherwise, show that

$$x^9 + 1 = Ax + B + (x^2 + 1)\,\psi(x),$$

and determine A, B, and $\psi(x)$.

3. Prove that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent, and that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

is convergent.

Show that in the last-named series the sum of the odd terms is three times the sum of the even terms.

4(*a*). If P and Q are each products of $2n$ consecutive numbers, prove that the difference between P and Q is divisible by the sum of the greatest and the least of all the $4n$ numbers.

4(*b*). If a and b are the first and last terms in a series of $(n+1)$ terms in arithmetical progression, show that the sum of the series of n terms obtained by multiplying each of the first n terms by the next consecutive term is $\frac{1}{3n}\{(n^2-1)(a^2+b^2)+(n^2+2)ab\}$.

5(*a*). Prove the binomial theorem for a positive integral index.

Prove that

$$2^{x-1}(1+2^x)=3^x+\frac{x(x-1)}{1.2}\cdot 3^{x-2}+\frac{x(x-1)(x-2)(x-3)}{1.2.3.4}\cdot 3^{x-4}+\ldots$$

5(*b*). Assuming the formulae

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,$$

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots,$$

deduce the identity

$$\log(n+1)-\log n=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\}.$$

Prove that

$$\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\ldots=\frac{1}{17}+\frac{1}{19}+\frac{1}{3}\left(\frac{1}{17^3}+\frac{1}{19^3}\right)+\ldots.$$

CENTRAL WELSH BOARD.

JULY, 1906. BLUE 2.

1. Show that

$$\frac{(d-b)(d-c)}{(a-b)(a-c)}+\frac{(d-c)(d-a)}{(b-c)(b-a)}+\frac{(d-a)(d-b)}{(c-a)(c-b)}=1.$$

2. Resolve into factors:

   (1) x^4-17x^2+16;

   (2) $a^4-b^4+2a^3b-2ab^3-4c^2(a^2-b^2)$.

3. Solve the equation $10x^2+31x-14=0$.

   Verify your solution graphically.

4. Solve the equations:

(1) $\dfrac{x-3a}{x+b}+\dfrac{x+2b}{x-a}=\dfrac{2x+b-2a}{x}$;

(2) $\dfrac{3}{x+2}+\dfrac{4}{x+1}+\dfrac{6}{x-3}=0$;

(3) $x^2+xy=12,\quad y^2-xy=8.$

5. If $a:b::c:d$, show that $pa^2+qb^2:pc^2+qd^2::ab:cd$.

Show that if to each of two numbers be added the number which is a mean proportional between them, two numbers are formed the square of whose ratio is the ratio of the original number.

6. (1) A man's salary starts at £a per annum and he receives a rise of £b every half-year for n years. How much will he have received at the end of $2n$ years?

(2) A geometric series consists of 6 terms, the first term is 27 and the last $3\frac{1}{3}$. Find its sum.

7. A man buys two horses for £70. By selling one of them at a profit of 20 per cent. and the other at a profit of 10 per cent. he gains altogether £10. What did he pay for each?

8. A man buys £10. 16s. worth of coal. If he had bought an inferior coal costing 3s. per ton less, he could for the same money have got one more ton. How many tons did he buy?

Senior Certificate. 1906. Blue 8.

1. The expression $x^4+ax^3+5x^2+bx+6$ when divided by $x-2$ leaves the remainder 16, and when divided by $x+1$ leaves the remainder 10. Find the values of a and b.

2. Resolve into factors:

(1) $bc(a^2+bc)-a(b^3+c^3)$;

(2) $x^4+2x^3y+x^2y^2-8y^3(x^2+xy)+12y^4$.

3. Show that the expression $4x^2-12x+11$ is always positive, and find the restrictions on the value of x that $4x^2-12x-7$ may be positive. Verify by drawing graphs.

4. If α and β are the roots of the equation

$$x^2+mx+m^2-2=0,$$

show that $\alpha^3-\beta^3=2(\alpha-\beta)$.

5. The first term of an Arithmetic Progression is a and the common difference d. Find its sum to n terms.

Find the sum to $2n$ terms of the series formed by writing down the former series but changing the signs of the 2nd, 4th, 6th, &c. terms.

6. If $\frac{a+x}{b+x}=\frac{b^n}{a^n}$ and $\frac{a+y}{b+y}=\frac{a^{n+1}}{b^{n+1}}$, show that, if a is not equal to b, $x:a::b:y$.

7. Find the square root of:

(1) $27-7\sqrt{5}$.

(2) $a^2+x^2+\sqrt{a^4+a^2x^2+x^4}$.

8. Solve the equations:

(1) $x^{\frac{3}{2}}-y^{\frac{3}{2}}=7(x^{\frac{1}{2}}-y^{\frac{1}{2}})$,

$x^{\frac{3}{2}}+y^{\frac{3}{2}}=3(x^{\frac{1}{2}}+y^{\frac{1}{2}})$.

(2) $(y+z)^2=a^2+x^2$,

$(z+x)^2=b^2+y^2$,

$(x+y)^2=c^2+z^2$.

9. Prove that $\log a+\log b=\log ab$.

If $\log_{10}2=x$, $\log_{10}3=y$, find in terms of x and y the logarithms of $\frac{3}{5}$, 75, ·0015.

JULY, 1906. GREEN 4.

1. If $f=x(y+z-x)$, $g=y(z+x-y)$, $h=z(x+y-z)$, show that $x:y:z::f(g+h-f):g(h+f-g):h(f+g-h)$.

Express also $2gh+2hf+2fg-f^2-g^2-h^2$ as the product of four factors linear in x, y, and z.

2. State and prove the rule for finding the greatest common measure of two quantities.

3. Solve the equations:

(1) $x+2y=3,\ x^3+x^2y+4xy^2+3y^3=9$;

(2) $\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2=14.$

4. Show that the expression $x+\frac{1}{x}+3$ can have no values for real values of x between 1 and 5. Verify this graphically.

5. In how many ways may 8 men be placed in a row, so that neither of two specified men may be at either end of the row?

6. Prove the binomial theorem for a positive integral index.

Find the coefficient of x^4 in $(1+x+2x^2)^6$.

7. Interpret, giving reasons, $a^{-\frac{p}{q}}$, where p and q are positive integers. If $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}=0$, show that $a^3+b^3+c^3=27abc$.

8. Prove that $\log a^m = m \log a$.

Given that $\log_{10}2 = \cdot30103$ and $\log_{10}3 = \cdot47712$, find the logarithms of $\cdot0015$ and 6750.

Find also the number of zeros following the decimal point in the value of $2^{\frac{1}{3}0} \div 3^7$, when expressed as a decimal.

9. Express 2253 in the scale of seven. Show that whatever be the value of n, a number will be divisible by $(n-1)$ if the sum of its digits, when expressed in the scale of n, is divisible by $n-1$.

10. Evaluate $\begin{vmatrix} 3 & -5 & 2 \\ -7 & 3 & 4 \\ 4 & 2 & -5 \end{vmatrix}$.

JULY, 1906. GREEN 5.

1. In how many ways may a council of 15 members, representing three parties, be constituted so that no party has an absolute majority?

2. Resolve into partial fractions $\frac{1}{(1-x)(1+x)^2(1+x^2)}$.

3. Solve the equations:

(1) $x(3-4x)^2+a(3-4a)^2=1$;

(2) $x(y+z+ayz) = y(z+x+bzx)$
$= z(x+y+cxy) = \frac{x+y+z}{a+b+c}$.

4. State and prove the law of the formation of the successive convergents to a continued fraction of the form

$$\frac{1}{a_1+}\ \frac{1}{a_2+}\ \frac{1}{a_3+}\ \cdots.$$

Show that the n^{th} convergent to the continued fraction

$$\frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+}\ \cdots$$

is equal to

$$\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\} \div \{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\}.$$

5. Show that the product of n positive quantities, whose sum is given, is greatest when they are all equal.

Point out where your proof would break down if some of the quantities might be negative.

6. Sum to n terms the series whose n^{th} terms are

(1) $n2^n$; (2) $n^3-n+1 \div n(n+1)$.

7. Expand $\log(1+x)$ in a series of ascending powers of x, examining the condition under which such an expansion is possible.

By writing $(1+x)^{\frac{1}{x}}$ in the form $e^{\frac{1}{x}\log(1+x)}$ (or otherwise), find the expansion of $(1+x)^{\frac{1}{x}}$ in ascending powers of x as far as the term containing x^2.

8. Show that the value of a determinant, which has two rows or columns identical, is zero.

Evaluate $\begin{vmatrix} (a+x)^2 & (a+y)^2 & (a+z)^2 \\ (b+x)^2 & (b+y)^2 & (b+z)^2 \\ (c+x)^2 & (c+y)^2 & (c+z)^2 \end{vmatrix}$.

9. Solve the equation $x^4-3x^3+7x^2-8x+6=0$, given that the sum of two of its roots is unity.

10. Show that between every successive pair of real roots of the equation $f(x)=0$, there is a real root of $f'(x)=0$.

Show that the equation $x^3+3x-5=0$ has only one real root, and find the value of that root correct to two places of decimals.

JULY, 1907. BLUE 2 A.

1. Find a value of a that will make $x^4+3x^3+4x^2+ax+1$ divisible by $x+1$.

Show that, with this value of a, it is divisible by $(x+1)^2$, and find the quotient.

2. Resolve into factors:

(1) a^3-8b^3; (2) $(xy+1)^4-(x-y)^4$.

3. Draw the graph of $y=x^2$ between the values -1.5 and $+2$ of x, taking the unit of x to be 4 cm., and the unit of y to be 2 cm.

Solve the equation $2x^2-5x+2=0$, and verify the solution by means of your graph.

4. Solve the equations:

(1) $\dfrac{x+1}{x+2}+\dfrac{x+2}{x+3}=\dfrac{4x-2}{2x+1}$;

(2) $x^3+3xy=10$, $x+y=3$.

5. If $a:b::c:d$, show that

$$pa+qb:pc+qd::ra+sb:rc+sd.$$

If $a:b::b:c$, show that

$$a-b:b::a-c:b+c.$$

6. Prove the formula for the sum of n terms of an Arithmetic Progression of which the first term is a and the common difference d.

If a, b, c, d are four consecutive terms of an Arithmetic Progression, show that $a^2-3b^2+3c^2-d^2=0$.

7. With £x a man buys equal amounts of a p per cent. stock at a, and a q per cent. stock at b. What is his income?

8. A, B, C all travel a certain distance; A and B are walking and B walks a mile an hour faster than A; C is bicycling and rides 8 miles an hour faster than B walks. The time taken by A is as much as that taken by B and C together. At what rate does A walk?

Additional Mathematics. July, 1907.

1. In *Whitaker's Almanack* for 1907 the Moon's Horizontal Parallax at Noon is given for the first fortnight of July by the following table:

July	1	57′ 32″.	July	8	54′ 2″.
,,	2	56′ 59″.	,,	9	53′ 57″.
,,	3	56′ 11″.	,,	10	53′ 57″.
,,	4	55′ 30″.	,,	11	54′ 3″.
,,	5	54′ 57″.	,,	12	54′ 14″.
,,	6	54′ 32″.	,,	13	54′ 31″.
,,	7	54′ 14″.	,,	14	54′ 53″.

Represent this table graphically, and find the Moon's Horizontal Parallax at midnight on July 14 and 6 p.m. on July 12.

2. Show that if the roots of the quadratic $ax^2+bx+c=0$ are imaginary, then the expression ax^2+bx+c has for all values of x the same sign as a.

If y is equal to $x(3-x)$ when x is less than 2, and to $7x-8-x^2$ when x is greater than 2, what are the greatest and least values that y takes as x increases from 1 to 4?

3. Solve:

(1) $xy=1$, $x^3-y^3=7\frac{7}{8}$;

(2) $\sqrt{4-x}\pm\sqrt{6+x}\pm\sqrt{3(x+4)}=0$.

Determine the ambiguities corresponding to the two roots.

4. If $a+\sqrt{b}=c+\sqrt{d}$, where a, b, c, d are rational and b and d are not perfect squares, show that $a=c$, $b=d$.

Find (1) The square root of $11+\sqrt{105}$;

(2) The cube root of $5\sqrt{2}-7$.

5. Draw the graph of $y=x^3$.

Employ it to show that the equation $x^3-3x+1=0$ has three real roots and to determine roughly their values.

6. Prove that $\log a^m = m\log a$.

If the increase in a tree's girth in one year is proportional to its girth at the beginning of the year and its girth is doubled in 11 years, in how many years will its girth be trebled?

7. Find by means of the Binomial Theorem the value, correct to 3 places of decimals, of the fifth root of 1·3. Verify your result by means of logarithms.

8. Find correct to 3 places of decimals the tenth root of the sum of the infinite series

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots.$$

9. The law connecting the pressure and volume of a given mass is $pv^\gamma = C$ where p is the pressure, v the volume and γ and C are constants; and the work done in compressing the given gas from a volume V to a volume U is $\frac{C}{\gamma-1}\left\{\frac{1}{U^{\gamma-1}}-\frac{1}{V^{\gamma-1}}\right\}$. If the gas is such that its pressure is doubled when its volume is compressed from 100 c.c. to 61 c.c., find as a numerical multiple of C the work done.

HIGHER MATHEMATICS. JULY, 1907. GREEN 1 A.

1. Resolve into factors:

(1) $a(b-c)^3+b(c-a)^3+c(a-b)^3$;

(2) $a^3+b^3+c^3-(a+b+c)(a^3-b^3+c^3)$.

2. If $\lambda(a_1x^2+2b_1x+c_1)+(a_2x^2+2b_2x+c_2)$ can be expressed in the form $\mu(x+a)^2$, show that

$$(a_1a-b_1)(b_2a-c_2)=(a_2a-b_2)(b_1a-c_1).$$

3. (1) Solve the equation

$$(x-7)(x-3)(x+1)(x+5)=1680;$$

(2) Find the cube root of $5\sqrt{2}-7$.

4. Draw the graph of the function $\frac{x^2+3}{x+1}$.

Show that, if x is real, $\frac{1}{2}>\frac{x+1}{x^2+3}>-\frac{1}{6}$.

5. In how many ways may seven persons distribute themselves among the ten seats of a compartment of a railway carriage, if three of them refuse to ride with their backs to the engine?

6. The sum of $n+1$ consecutive terms $a_0, a_1, a_2, \ldots a_n$ of an Arithmetical Progression is s_1 and $c_0, c_1, c_2 \ldots c_n$ are the successive coefficients in the expansion of $(1+x)^n$; show that

$$(n+1)(c_0a_0+c_1a_1+c_2a_2+\ldots+c_na_n) = 2^n s.$$

7. Show that if m and n are positive integers ($m > n$)

$$a^m \times a^n = a^{m+n}, \quad a^m \div a^n = a^{m-n}, \quad (a^m)^n = a^{mn}.$$

Deduce interpretations of a^0, $a^{-\frac{p}{q}}$.

If $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} = 0$, show that

$$(a^2+b^2+c^2+d^2-2bc-2ca-2ab-2ad-2bd-2cd)^2 = 64abcd.$$

8. Prove that $\log ab = \log a + \log b$.

Find, from tables, the value of $(2.3)^{\frac{2}{3}}$, $\sqrt[5]{.0024}$.

9. Multiply together 231 and 45 in the scale of 8.

Verify your result by transforming factors and product into the scale of 10.

10. Solve by means of determinants:

$$2x-5y+3z = 6, \quad x+y+z = 8, \quad 4x+2y-z = -3.$$

HIGHER MATHEMATICS. 1907. GREEN 2 A.

1. If $f(x) \equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$, show that, when $f(x)$ is divided by $x-\alpha$, the remainder is $f(\alpha)$ and that when $f(x)$ is divided by $(x-\alpha)(x-\beta)$, the remainder is

$$\frac{x\{f(\alpha)-f(\beta)\}+\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta}.$$

2. Solve:

(1) $(x-7)(x-3)(x+1)(x+5) = 1680$;

(2) $x^2+2yz = 3a^2-2b^2$, $y^2+2zx = 3a^2+b^2$,

$$yz+zx+xy = 3a^2-b^2.$$

3. Show that the coefficient of x^{n-1} in the expansion of $(1-x)^{-(m+1)}$, where m is a positive integer, is

$$\frac{n(n+1)(n+2)\ldots(n+m-1)}{\lfloor m}.$$

Determine a, b, c, d, so that the n^{th} term in the expansion of

$$\frac{a+bx+cx^2+dx^3}{(1-x)^4}$$

may be n^3x^{n-1}.

4. Find the sum to n terms of the series whose n^{th} terms are

(1) n^3, (2) $\dfrac{nr^n}{\lfloor n+r}$.

5. Show that any convergent to the continued fraction

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,$$

where $a_1\, a_2\, a_3$ are all positive integers, is nearer to the value of the fraction than any fraction which has a smaller denominator than the given convergent.

Convert $\sqrt{33}$ into a recurring continued fraction.

6. If $E(x)$ denotes the infinite series $1+x+\dfrac{x^2}{\lfloor 2}+\dfrac{x^3}{\lfloor 3}+\ldots$,

(1) Show that $E(x)$ is convergent for all values of x;

(2) Show by actual multiplication that

$$E(x)\times E(y) = E(x+y);$$

(3) Deduce that for all real values of x,

$$E(x) = \{E(1)\}^x.$$

7. Show that

$$\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$$

8. Solve the cubic $x^3+3px+q=0$ by Cardan's method, and show that if its roots are all real p^3+4q^2 is negative.

Find correct to two decimal places the real root of

$$x^3-3x-5=0.$$

9. (1) Show that whatever be the value of k the equation

$$(x^2-1)^2-kx\,(x^3+1)=0$$

has two real and two imaginary roots.

(2) If α is a root, not unity, of the equation $x^7-1=0$, show that $\alpha+\alpha^2+\alpha^4$ is a root of the equation $x^3+x+2=0$.